4 □ APPLICATIONS OF DIFFERENTIATION

4.1 Maximum and Minimum Values

1. A function f has an **absolute minimum** at $x = c$ if $f(c)$ is the smallest function value on the entire domain of f, whereas f has a **local minimum** at c if $f(c)$ is the smallest function value when x is near c.

2. (a) The Extreme Value Theorem

(b) See the Closed Interval Method.

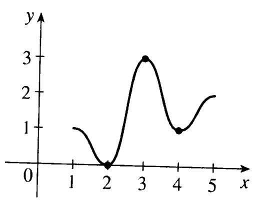
3. Absolute maximum at b; absolute minimum at d; local maxima at b and e; local minima at d and s; neither a maximum nor a minimum at a, c, r, and t.

4. Absolute maximum at e; absolute minimum at t; local maxima at c, e, and s; local minima at b, c, d, and r; neither a maximum nor a minimum at a.

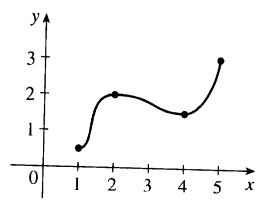
5. Absolute maximum value is $f(4) = 4$; absolute minimum value is $f(7) = 0$; local maximum values are $f(4) = 4$ and $f(6) = 3$; local minimum values are $f(2) = 1$ and $f(5) = 2$.

6. Absolute maximum value is $f(8) = 5$; absolute minimum value is $f(2) = 0$; local maximum values are $f(1) = 2$, $f(4) = 4$, and $f(6) = 3$; local minimum values are $f(2) = 0$, $f(5) = 2$, and $f(7) = 1$.

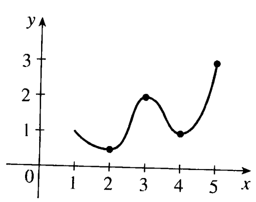
7. Absolute minimum at 2, absolute maximum at 3, local minimum at 4



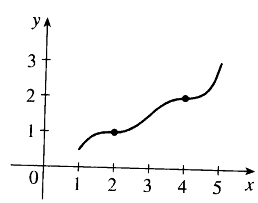
8. Absolute minimum at 1, absolute maximum at 5, local maximum at 2, local minimum at 4



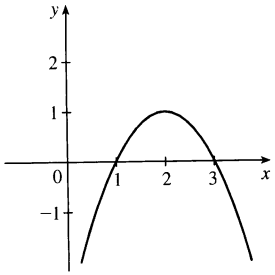
9. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4



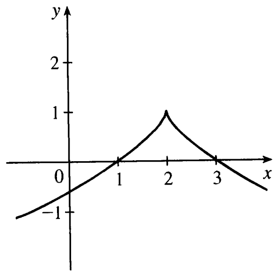
10. f has no local maximum or minimum, but 2 and 4 are critical numbers



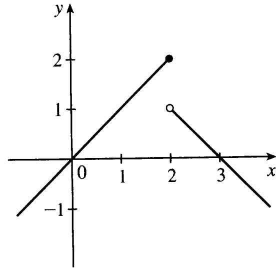
11. (a)



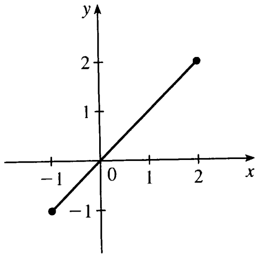
(b)



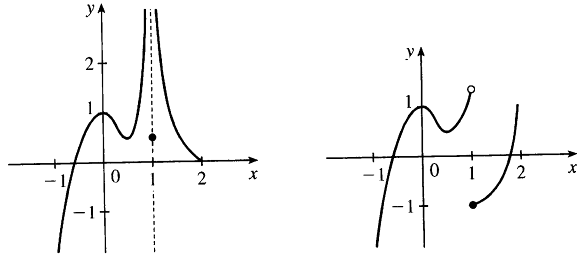
(c)



12. (a) Note that a local maximum cannot occur at an endpoint.

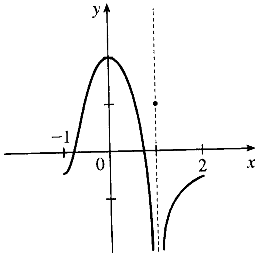


(b)

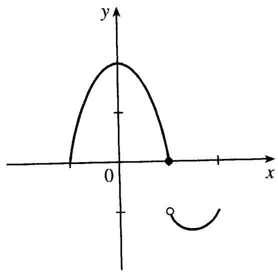


Note: By the Extreme Value Theorem, f must *not* be continuous.

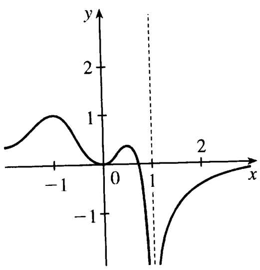
13. (a) *Note:* By the Extreme Value Theorem, f must *not* be continuous; because if it were, it would attain an absolute minimum.



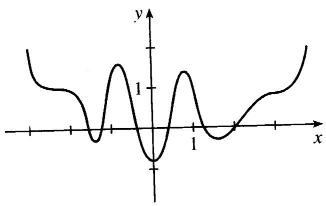
(b)



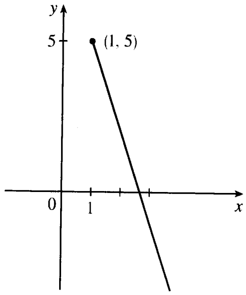
14. (a)



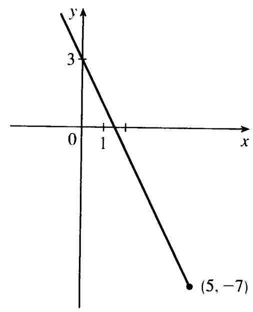
(b)



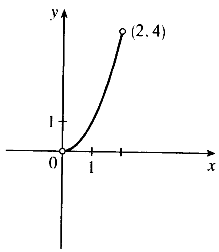
15. $f(x) = 8 - 3x$, $x \geq 1$. Absolute maximum $f(1) = 5$; no local maximum. No absolute or local minimum.



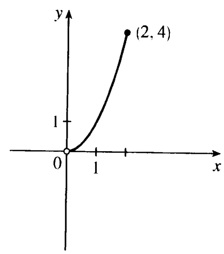
16. $f(x) = 3 - 2x$, $x \leq 5$. Absolute minimum $f(5) = -7$; no local minimum. No absolute or local maximum.



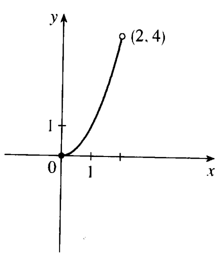
17. $f(x) = x^2$, $0 < x < 2$. No absolute or local maximum or minimum value.



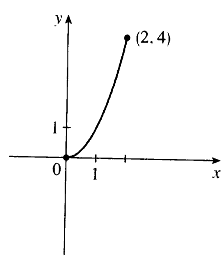
18. $f(x) = x^2$, $0 < x \leq 2$. Absolute maximum $f(2) = 4$; no local maximum. No absolute or local minimum.



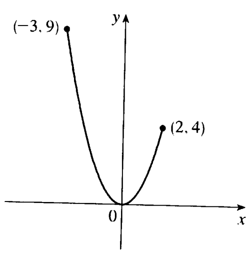
19. $f(x) = x^2$, $0 \leq x < 2$. Absolute minimum $f(0) = 0$; no local minimum. No absolute or local maximum.



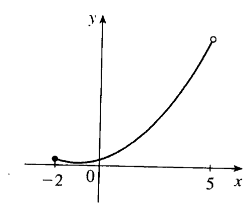
20. $f(x) = x^2$, $0 \leq x \leq 2$. Absolute maximum $f(2) = 4$. Absolute minimum $f(0) = 0$. No local maximum or minimum.



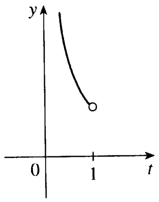
21. $f(x) = x^2$, $-3 \leq x \leq 2$. Absolute maximum $f(-3) = 9$. No local maximum. Absolute and local minimum $f(0) = 0$.



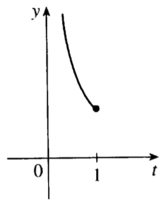
22. $f(x) = 1 + (x + 1)^2$, $-2 \leq x < 5$. No absolute or local maximum. Absolute and local minimum $f(-1) = 1$.



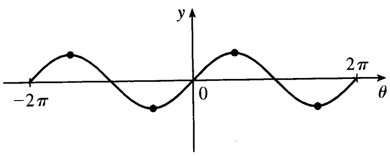
23. $f(t) = 1/t$, $0 < t < 1$. No maximum or minimum.



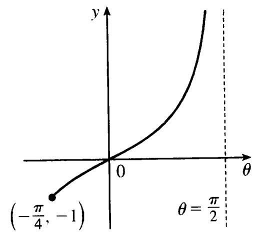
24. $f(t) = 1/t$, $0 < t \le 1$. Absolute minimum $f(1) = 1$; no local minimum. No local or absolute maximum.



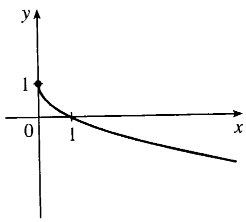
25. $f(\theta) = \sin\theta$, $-2\pi \le \theta \le 2\pi$. Absolute and local maxima $f\left(-\frac{3\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = 1$. Absolute and local minima $f\left(-\frac{\pi}{2}\right) = f\left(\frac{3\pi}{2}\right) = -1$.



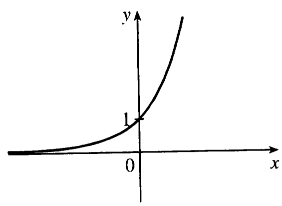
26. $f(\theta) = \tan\theta$, $-\frac{\pi}{4} \le \theta < \frac{\pi}{2}$. Absolute minimum $f\left(-\frac{\pi}{4}\right) = -1$; no local minimum. No absolute or local maximum.



27. $f(x) = 1 - \sqrt{x}$. Absolute maximum $f(0) = 1$; no local maximum. No absolute or local minimum.

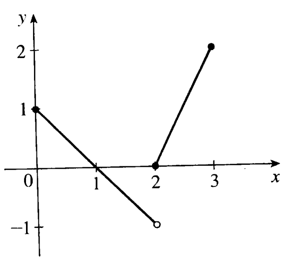


28. $f(x) = e^x$. No absolute or local maximum or minimum value.



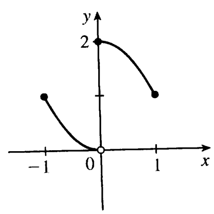
29. $f(x) = \begin{cases} 1 - x & \text{if } 0 \le x < 2 \\ 2x - 4 & \text{if } 2 \le x \le 3 \end{cases}$

Absolute maximum $f(3) = 2$; no local maximum. No absolute or local minimum.



30. $f(x) = \begin{cases} x^2 & \text{if } -1 \le x < 0 \\ 2 - x^2 & \text{if } 0 \le x \le 1 \end{cases}$

Absolute and local maximum $f(0) = 2$. No absolute or local minimum.



31. $f(x) = 5x^2 + 4x \Rightarrow f'(x) = 10x + 4$. $f'(x) = 0 \Rightarrow x = -\frac{2}{5}$, so $-\frac{2}{5}$ is the only critical number.

32. $f(x) = x^3 + x^2 - x \Rightarrow f'(x) = 3x^2 + 2x - 1$. $f'(x) = 0 \Rightarrow (x+1)(3x-1) = 0 \Rightarrow x = -1, \frac{1}{3}$. These are the only critical numbers.

33. $f(x) = x^3 + 3x^2 - 24x \Rightarrow f'(x) = 3x^2 + 6x - 24 = 3(x^2 + 2x - 8)$.
$f'(x) = 0 \Rightarrow 3(x+4)(x-2) = 0 \Rightarrow x = -4, 2$. These are the only critical numbers.

34. $f(x) = x^3 + x^2 + x \Rightarrow f'(x) = 3x^2 + 2x + 1$. $f'(x) = 0 \Rightarrow 3x^2 + 2x + 1 = 0 \Rightarrow$
$x = \dfrac{-2 \pm \sqrt{4 - 12}}{6}$. Neither of these is a real number. Thus, there are no critical numbers.

35. $s(t) = 3t^4 + 4t^3 - 6t^2 \Rightarrow s'(t) = 12t^3 + 12t^2 - 12t$. $s'(t) = 0 \Rightarrow 12t(t^2 + t - 1) \Rightarrow t = 0$ or $t^2 + t - 1 = 0$. Using the quadratic formula to solve the latter equation gives us
$t = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \dfrac{-1 \pm \sqrt{5}}{2} \approx 0.618, -1.618$. The three critical numbers are $0, \dfrac{-1 \pm \sqrt{5}}{2}$.

36. $f(z) = \dfrac{z+1}{z^2 + z + 1} \Rightarrow f'(z) = \dfrac{(z^2 + z + 1)\,1 - (z+1)(2z+1)}{(z^2 + z + 1)^2} = \dfrac{-z^2 - 2z}{(z^2 + z + 1)^2} = 0 \Leftrightarrow$
$z(z+2) = 0 \Rightarrow z = 0, -2$ are the critical numbers. (Note that $z^2 + z + 1 \neq 0$ since the discriminant < 0.)

37. $g(x) = |2x + 3| = \begin{cases} 2x + 3 & \text{if } 2x + 3 \geq 0 \\ -(2x+3) & \text{if } 2x + 3 < 0 \end{cases} \Rightarrow g'(x) = \begin{cases} 2 & \text{if } x > -\frac{3}{2} \\ -2 & \text{if } x < -\frac{3}{2} \end{cases}$
$g'(x)$ is never 0, but $g'(x)$ does not exist for $x = -\frac{3}{2}$, so $-\frac{3}{2}$ is the only critical number.

38. $g(x) = x^{1/3} - x^{-2/3} \Rightarrow g'(x) = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} = \frac{1}{3}x^{-5/3}(x+2) = \dfrac{x+2}{3x^{5/3}}$.
$g'(-2) = 0$ and $g'(0)$ does not exist, but 0 is not in the domain of g, so the only critical number is -2.

39. $g(t) = 5t^{2/3} + t^{5/3} \Rightarrow g'(t) = \frac{10}{3}t^{-1/3} + \frac{5}{3}t^{2/3}$. $g'(0)$ does not exist, so $t = 0$ is a critical number.
$g'(t) = \frac{5}{3}t^{-1/3}(2 + t) = 0 \Leftrightarrow t = -2$, so $t = -2$ is also a critical number.

40. $g(t) = \sqrt{t}\,(1 - t) = t^{1/2} - t^{3/2} \Rightarrow g'(t) = \dfrac{1}{2\sqrt{t}} - \frac{3}{2}\sqrt{t}$. $g'(0)$ does not exist, so $t = 0$ is a critical number.
$0 = g'(t) = \dfrac{1 - 3t}{2\sqrt{t}} \Rightarrow t = \frac{1}{3}$, so $t = \frac{1}{3}$ is also a critical number.

41. $F(x) = x^{4/5}(x-4)^2 \Rightarrow$
$$F'(x) = x^{4/5} \cdot 2(x-4) + (x-4)^2 \cdot \tfrac{4}{5}x^{-1/5} = \tfrac{1}{5}x^{-1/5}(x-4)\,[5 \cdot x \cdot 2 + (x-4) \cdot 4]$$
$$= \frac{(x-4)(14x - 16)}{5x^{1/5}} = \frac{2(x-4)(7x-8)}{5x^{1/5}} = 0 \text{ when } x = 4, \tfrac{8}{7}\text{; and } F'(0) \text{ does not exist.}$$
Critical numbers are $0, \frac{8}{7}, 4$.

42. $G(x) = \sqrt[3]{x^2 - x} \Rightarrow G'(x) = \frac{1}{3}(x^2 - x)^{-2/3}(2x - 1)$. $G'(x)$ does not exist when $x^2 - x = 0$, that is, when $x = 0$ or 1. $G'(x) = 0 \Leftrightarrow 2x - 1 = 0 \Leftrightarrow x = \frac{1}{2}$. So the critical numbers are $x = 0, \frac{1}{2}, 1$.

43. $f(\theta) = 2\cos\theta + \sin^2\theta \Rightarrow f'(\theta) = -2\sin\theta + 2\sin\theta\cos\theta$. $f'(\theta) = 0 \Rightarrow 2\sin\theta(\cos\theta - 1) = 0 \Rightarrow$ $\sin\theta = 0$ or $\cos\theta = 1 \Rightarrow \theta = n\pi$ (n an integer) or $\theta = 2n\pi$. The solutions $\theta = n\pi$ include the solutions $\theta = 2n\pi$, so the critical numbers are $\theta = n\pi$.

44. $g(\theta) = 4\theta - \tan\theta \Rightarrow g'(\theta) = 4 - \sec^2\theta$. $g'(\theta) = 0 \Rightarrow \sec^2\theta = 4 \Rightarrow \sec\theta = \pm 2 \Rightarrow$ $\cos\theta = \pm\frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, \frac{2\pi}{3} + 2n\pi$, and $\frac{4\pi}{3} + 2n\pi$ are critical numbers.

Note: The values of θ that make $g'(\theta)$ undefined are not in the domain of g.

45. $f(x) = x\ln x \Rightarrow f'(x) = x(1/x) + (\ln x)\cdot 1 = \ln x + 1$. $f'(x) = 0 \Leftrightarrow \ln x = -1 \Leftrightarrow$ $x = e^{-1} = 1/e$. Therefore, the only critical number is $x = 1/e$.

46. $f(x) = xe^{2x} \Rightarrow f'(x) = x(2e^{2x}) + e^{2x} = e^{2x}(2x+1)$. Since e^{2x} is never 0, we have $f'(x) = 0$ only when $2x + 1 = 0 \Leftrightarrow x = -\frac{1}{2}$. So $-\frac{1}{2}$ is the only critical number.

47. $f(x) = 3x^2 - 12x + 5$, $[0, 3]$. $f'(x) = 6x - 12 = 0 \Leftrightarrow x = 2$. Applying the Closed Interval Method, we find that $f(0) = 5$, $f(2) = -7$, and $f(3) = -4$. So $f(0) = 5$ is the absolute maximum value and $f(2) = -7$ is the absolute minimum value.

48. $f(x) = x^3 - 3x + 1$, $[0, 3]$. $f'(x) = 3x^2 - 3 = 0 \Leftrightarrow x = \pm 1$, but -1 is not in $[0, 3]$. $f(0) = 1$, $f(1) = -1$, and $f(3) = 19$. So $f(3) = 19$ is the absolute maximum value and $f(1) = -1$ is the absolute minimum value.

49. $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$. $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1) = 0 \Leftrightarrow$ $x = 2, -1$. $f(-2) = -3$, $f(-1) = 8$, $f(2) = -19$, and $f(3) = -8$. So $f(-1) = 8$ is the absolute maximum value and $f(2) = -19$ is the absolute minimum value.

50. $f(x) = x^3 - 6x^2 + 9x + 2$, $[-1, 4]$. $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3) = 0 \Leftrightarrow$ $x = 1, 3$. $f(-1) = -14$, $f(1) = 6$, $f(3) = 2$, and $f(4) = 6$. So $f(1) = f(4) = 6$ is the absolute maximum value and $f(-1) = -14$ is the absolute minimum value.

51. $f(x) = x^4 - 2x^2 + 3$, $[-2, 3]$. $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1) = 0 \Leftrightarrow x = -1, 0, 1$. $f(-2) = 11$, $f(-1) = 2$, $f(0) = 3$, $f(1) = 2$, $f(3) = 66$. So $f(3) = 66$ is the absolute maximum value and $f(\pm 1) = 2$ is the absolute minimum value.

52. $f(x) = (x^2 - 1)^3$, $[-1, 2]$. $f'(x) = 3(x^2 - 1)^2(2x) = 6x(x+1)^2(x-1)^2 = 0 \Leftrightarrow x = -1, 0, 1$. $f(\pm 1) = 0$, $f(0) = -1$, and $f(2) = 27$. So $f(2) = 27$ is the absolute maximum value and $f(0) = -1$ is the absolute minimum value.

53. $f(x) = \dfrac{x}{x^2+1}$, $[0, 2]$. $f'(x) = \dfrac{(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1 - x^2}{(x^2+1)^2} = 0 \Leftrightarrow x = \pm 1$, but -1 is not in $[0, 2]$. $f(0) = 0$, $f(1) = \frac{1}{2}$, $f(2) = \frac{2}{5}$. So $f(1) = \frac{1}{2}$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

54. $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$, $[-4, 4]$. $f'(x) = \dfrac{(x^2+4)(2x) - (x^2-4)(2x)}{(x^2+4)^2} = \dfrac{16x}{(x^2+4)^2} = 0 \Leftrightarrow x = 0$. $f(\pm 4) = \frac{12}{20} = \frac{3}{5}$ and $f(0) = -1$. So $f(\pm 4) = \frac{3}{5}$ is the absolute maximum value and $f(0) = -1$ is the absolute minimum value.

55. $f(t) = t\sqrt{4-t^2}$, $[-1, 2]$.

$$f'(t) = t \cdot \tfrac{1}{2}\left(4-t^2\right)^{-1/2}(-2t) + \left(4-t^2\right)^{1/2} \cdot 1 = \frac{-t^2}{\sqrt{4-t^2}} + \sqrt{4-t^2} = \frac{-t^2 + \left(4-t^2\right)}{\sqrt{4-t^2}} = \frac{4-2t^2}{\sqrt{4-t^2}}.$$

$f'(t) = 0 \;\Rightarrow\; 4 - 2t^2 = 0 \;\Rightarrow\; t^2 = 2 \;\Rightarrow\; t = \pm\sqrt{2}$, but $t = -\sqrt{2}$ is not in the given interval, $[-1, 2]$. $f'(t)$ does not exist if $4 - t^2 = 0 \;\Rightarrow\; t = \pm 2$, but -2 is not in the given interval. $f(-1) = -\sqrt{3}$, $f\left(\sqrt{2}\right) = 2$, and $f(2) = 0$. So $f\left(\sqrt{2}\right) = 2$ is the absolute maximum value and $f(-1) = -\sqrt{3}$ is the absolute minimum value.

56. $f(t) = \sqrt[3]{t}\,(8-t)$, $[0, 8]$. $f(t) = 8t^{1/3} - t^{4/3} \;\Rightarrow\; f'(t) = \frac{8}{3}t^{-2/3} - \frac{4}{3}t^{1/3} = \frac{4}{3}t^{-2/3}(2-t) = \dfrac{4(2-t)}{3\sqrt[3]{t^2}}$.

$f'(t) = 0 \;\Rightarrow\; t = 2$. $f'(t)$ does not exist if $t = 0$. $f(0) = 0$, $f(2) = 6\sqrt[3]{2} \approx 7.56$, and $f(8) = 0$. So $f(2) = 6\sqrt[3]{2}$ is the absolute maximum value and $f(0) = f(8) = 0$ is the absolute minimum value.

57. $f(x) = \sin x + \cos x$, $\left[0, \frac{\pi}{3}\right]$. $f'(x) = \cos x - \sin x = 0 \;\Leftrightarrow\; \sin x = \cos x \;\Rightarrow\; \dfrac{\sin x}{\cos x} = 1 \;\Rightarrow$

$\tan x = 1 \;\Rightarrow\; x = \frac{\pi}{4}$. $f(0) = 1$, $f\left(\frac{\pi}{4}\right) = \sqrt{2} \approx 1.41$, $f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}+1}{2} \approx 1.37$. So $f\left(\frac{\pi}{4}\right) = \sqrt{2}$ is the absolute maximum value and $f(0) = 1$ is the absolute minimum value.

58. $f(x) = x - 2\cos x$, $[-\pi, \pi]$. $f'(x) = 1 + 2\sin x = 0 \;\Leftrightarrow\; \sin x = -\frac{1}{2} \;\Leftrightarrow\; x = -\frac{5\pi}{6}, -\frac{\pi}{6}$.

$f(-\pi) = 2 - \pi \approx -1.14$, $f\left(-\frac{5\pi}{6}\right) = \sqrt{3} - \frac{5\pi}{6} \approx -0.886$, $f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - \sqrt{3} \approx -2.26$, $f(\pi) = \pi + 2 \approx 5.14$. So $f(\pi) = \pi + 2$ is the absolute maximum value and $f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - \sqrt{3}$ is the absolute minimum value.

59. $f(x) = xe^{-x}$, $[0, 2]$. $f'(x) = x(-e^{-x}) + e^{-x} = e^{-x}(1-x) = 0 \;\Leftrightarrow\; x = 1$.

$f(0) = 0$, $f(1) = e^{-1} = 1/e \approx 0.37$, $f(2) = 2/e^2 \approx 0.27$. So $f(1) = 1/e$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

60. $f(x) = \dfrac{\ln x}{x}$, $[1, 3]$. $f'(x) = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2} = 0 \;\Leftrightarrow\; 1 - \ln x = 0 \;\Leftrightarrow\; \ln x = 1 \;\Leftrightarrow\; x = e$.

$f(1) = 0/1 = 0$, $f(e) = 1/e \approx 0.368$, $f(3) = (\ln 3)/3 \approx 0.366$. So $f(e) = 1/e$ is the absolute maximum value and $f(1) = 0$ is the absolute minimum value.

61. $f(x) = x - 3\ln x$, $[1, 4]$. $f'(x) = 1 - \dfrac{3}{x} = \dfrac{x-3}{x} = 0 \;\Leftrightarrow\; x = 3$. f' does not exist for $x = 0$, but 0 is not in the domain of f. $f(1) = 1$, $f(3) = 3 - 3\ln 3 \approx -0.296$, $f(4) = 4 - 3\ln 4 \approx -0.159$. So $f(1) = 1$ is the absolute maximum value and $f(3) = 3 - 3\ln 3 \approx -0.296$ is the absolute minimum value.

62. $f(x) = e^{-x} - e^{-2x}$, $[0, 1]$. $f'(x) = e^{-x}(-1) - e^{-2x}(-2) = \dfrac{2}{e^{2x}} - \dfrac{1}{e^x} = \dfrac{2 - e^x}{e^{2x}} = 0 \;\Leftrightarrow\; e^x = 2 \;\Leftrightarrow$

$x = \ln 2 \approx 0.69$. $f(0) = 0$, $f(\ln 2) = e^{-\ln 2} - e^{-2\ln 2} = \left(e^{\ln 2}\right)^{-1} - \left(e^{\ln 2}\right)^{-2} = 2^{-1} - 2^{-2} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. $f(1) = e^{-1} - e^{-2} \approx 0.233$. So $f(\ln 2) = \frac{1}{4}$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

63. $f(x) = x^a(1-x)^b$, $0 \le x \le 1$, $a > 0$, $b > 0$.

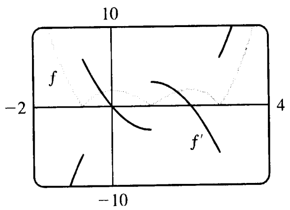
$$f'(x) = x^a \cdot b(1-x)^{b-1}(-1) + (1-x)^b \cdot ax^{a-1} = x^{a-1}(1-x)^{b-1}\left[x \cdot b(-1) + (1-x)\cdot a\right]$$
$$= x^{a-1}(1-x)^{b-1}(a - ax - bx)$$

At the endpoints, we have $f(0) = f(1) = 0$ [the minimum value of f]. In the interval $(0,1)$, $f'(x) = 0 \Leftrightarrow$ $x = \dfrac{a}{a+b}$.

$$f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a\left(1 - \frac{a}{a+b}\right)^b = \frac{a^a}{(a+b)^a}\left(\frac{a+b-a}{a+b}\right)^b = \frac{a^a}{(a+b)^a}\cdot\frac{b^b}{(a+b)^b} = \frac{a^a b^b}{(a+b)^{a+b}}.$$

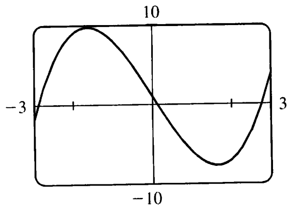
So $f\left(\dfrac{a}{a+b}\right) = \dfrac{a^a b^b}{(a+b)^{a+b}}$ is the absolute maximum value.

64.



We see that $f'(x) = 0$ at about $x = 0.0$ and 2.0, and that $f'(x)$ does not exist at about $x = -0.7$, 1.0, and 2.7, so the critical numbers of f are about -0.7, 0.0, 1.0, 2.0, and 2.7.

65. (a)



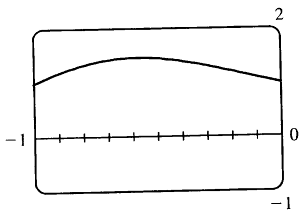
From the graph, it appears that the absolute maximum value is about $f(-1.63) = 9.71$, and the absolute minimum value is about $f(1.63) = -7.71$. These values make sense because the graph is symmetric about the point $(0,1)$. ($y = x^3 - 8x$ is symmetric about the origin.)

(b) $f(x) = x^3 - 8x + 1 \;\Rightarrow\; f'(x) = 3x^2 - 8$. So $f'(x) = 0 \;\Rightarrow\; x = \pm\sqrt{\frac{8}{3}}$.

$$f\left(\pm\sqrt{\tfrac{8}{3}}\right) = \left(\pm\sqrt{\tfrac{8}{3}}\right)^3 - 8\left(\pm\sqrt{\tfrac{8}{3}}\right) + 1 = \pm\tfrac{8}{3}\sqrt{\tfrac{8}{3}} \mp 8\sqrt{\tfrac{8}{3}} + 1$$
$$= -\tfrac{16}{3}\sqrt{\tfrac{8}{3}} + 1 = 1 - \tfrac{32\sqrt{6}}{9} \text{ [minimum] or } \tfrac{16}{3}\sqrt{\tfrac{8}{3}} + 1 = 1 + \tfrac{32\sqrt{6}}{9} \text{ [maximum]}$$

(From the graph, we see that the extreme values do not occur at the endpoints.)

66. (a)

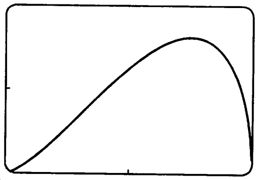


From the graph, it appears that the absolute maximum value is about $f(-0.58) = 1.47$, and the absolute minimum value is about $f(-1) = f(0) = 1.00$; that is, at both endpoints.

(b) $f(x) = e^{x^3 - x} \;\Rightarrow\; f'(x) = e^{x^3 - x}(3x^2 - 1)$. So $f'(x) = 0$ on $[-1, 0] \;\Rightarrow\; x = -\sqrt{1/3}$.

$f(-1) = f(0) = 1$ (minima) and $f\left(-\sqrt{1/3}\right) = e^{-\sqrt{3}/9 + \sqrt{3}/3} = e^{2\sqrt{3}/9}$ (maximum).

67. (a)



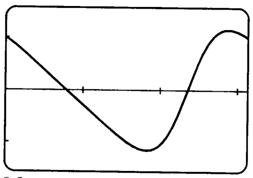
From the graph, it appears that the absolute maximum value is about $f(0.75) = 0.32$, and the absolute minimum value is $f(0) = f(1) = 0$; that is, at both endpoints.

(b) $f(x) = x\sqrt{x - x^2} \quad \Rightarrow \quad f'(x) = x \cdot \dfrac{1 - 2x}{2\sqrt{x - x^2}} + \sqrt{x - x^2} = \dfrac{(x - 2x^2) + (2x - 2x^2)}{2\sqrt{x - x^2}} = \dfrac{3x - 4x^2}{2\sqrt{x - x^2}}.$

So $f'(x) = 0 \quad \Rightarrow \quad 3x - 4x^2 = 0 \quad \Rightarrow \quad x(3 - 4x) = 0 \quad \Rightarrow \quad x = 0$ or $\frac{3}{4}$. $f(0) = f(1) = 0$ [minimum],

and $f\left(\frac{3}{4}\right) = \frac{3}{4}\sqrt{\frac{3}{4} - \left(\frac{3}{4}\right)^2} = \frac{3\sqrt{3}}{16}$ [maximum].

68. (a)



From the graph, it appears that the absolute maximum value is about $f(5.76) = 0.58$, and the absolute minimum value is about $f(3.67) = -0.58$.

(b) $f(x) = \dfrac{\cos x}{2 + \sin x} \quad \Rightarrow \quad f'(x) = \dfrac{(2 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(2 + \sin x)^2} = \dfrac{-1 - 2\sin x}{(2 + \sin x)^2}.$

So $f'(x) = 0 \quad \Rightarrow \quad \sin x = -\frac{1}{2} \quad \Rightarrow \quad x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$. Now $f\left(\frac{7\pi}{6}\right) = \frac{-\sqrt{3}/2}{3/2} = -\frac{1}{\sqrt{3}}$ [minimum],

and $f\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}/2}{3/2} = \frac{1}{\sqrt{3}}$ [maximum].

69. The density is defined as $\rho = \dfrac{\text{mass}}{\text{volume}} = \dfrac{1000}{V(T)}$ (in g/cm^3). But a critical point of ρ will also be a critical point of V $\left[\text{since } \dfrac{d\rho}{dT} = -1000V^{-2}\dfrac{dV}{dT} \text{ and } V \text{ is never } 0\right]$, and V is easier to differentiate than ρ.

$V(T) = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3 \quad \Rightarrow$

$V'(T) = -0.06426 + 0.0170086T - 0.0002037T^2$. Setting this equal to 0 and using the quadratic formula to find T, we get $T = \dfrac{-0.0170086 \pm \sqrt{0.0170086^2 - 4 \cdot 0.0002037 \cdot 0.06426}}{2(-0.0002037)} \approx 3.9665°\text{C}$ or $79.5318°\text{C}$. Since we are only interested in the region $0°\text{C} \leq T \leq 30°\text{C}$, we check the density ρ at the endpoints and at $3.9665°\text{C}$:

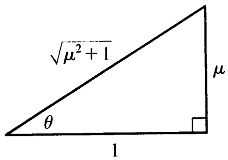
$\rho(0) \approx \dfrac{1000}{999.87} \approx 1.00013$; $\rho(30) \approx \dfrac{1000}{1003.7628} \approx 0.99625$; $\rho(3.9665) \approx \dfrac{1000}{999.7447} \approx 1.000255$. So water has its maximum density at about $3.9665°\text{C}$.

70. $F = \dfrac{\mu W}{\mu \sin\theta + \cos\theta} \Rightarrow \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta+\cos\theta)^2} = \dfrac{-\mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta+\cos\theta)^2}$.

So $\dfrac{dF}{d\theta} = 0 \Rightarrow \mu\cos\theta - \sin\theta = 0 \Rightarrow \mu = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$. Substituting $\tan\theta$ for μ in F gives us

$$F = \frac{(\tan\theta)W}{(\tan\theta)\sin\theta + \cos\theta} = \frac{W\tan\theta}{\dfrac{\sin^2\theta}{\cos\theta} + \cos\theta} = \frac{W\tan\theta\cos\theta}{\sin^2\theta + \cos^2\theta} = \frac{W\sin\theta}{1} = W\sin\theta.$$

If $\tan\theta = \mu$, then $\sin\theta = \dfrac{\mu}{\sqrt{\mu^2+1}}$ (see the figure), so $F = \dfrac{\mu}{\sqrt{\mu^2+1}}W$. We compare this with the value of F at the endpoints: $F(0) = \mu W$ and $F\left(\frac{\pi}{2}\right) = W$.

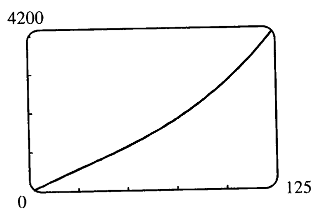


Now because $\dfrac{\mu}{\sqrt{\mu^2+1}} \le 1$ and $\dfrac{\mu}{\sqrt{\mu^2+1}} \le \mu$, we have that $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is less than or equal to each of $F(0)$ and $F\left(\frac{\pi}{2}\right)$. Hence, $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is the absolute minimum value of $F(\theta)$, and it occurs when $\tan\theta = \mu$.

71. We apply the Closed Interval Method to the continuous function

$I(t) = 0.00009045t^5 + 0.001438t^4 - 0.06561t^3 + 0.4598t^2 - 0.6270t + 99.33$ on $[0, 10]$. Its derivative is $I'(t) = 0.00045225t^4 + 0.005752t^3 - 0.19683t^2 + 0.9196t - 0.6270$. Since I' exists for all t, the only critical numbers of I occur when $I'(t) = 0$. We use a root-finder on a computer algebra system (or a graphing device) to find that $I'(t) = 0$ when $t \approx -29.7186, 0.8231, 5.1309$, or 11.0459, but only the second and third roots lie in the interval $[0, 10]$. The values of I at these critical numbers are $I(0.8231) \approx 99.09$ and $I(5.1309) \approx 100.67$. The values of I at the endpoints of the interval are $I(0) = 99.33$ and $I(10) \approx 96.86$. Comparing these four numbers, we see that food was most expensive at $t \approx 5.1309$ (corresponding roughly to August, 1989) and cheapest at $t = 10$ (midyear 1994).

72. (a)



The equation of the graph in the figure is

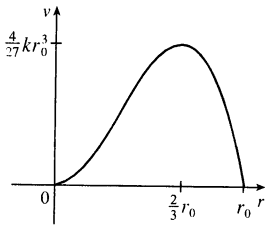
$v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872$.

(b) $a(t) = v'(t) = 0.00438t^2 - 0.23106t + 24.98169 \Rightarrow a'(t) = 0.00876t - 0.23106$. $a'(t) = 0 \Rightarrow$ $t_1 = \frac{0.23106}{0.00876} \approx 26.4$. $a(0) \approx 24.98$, $a(t_1) \approx 21.93$, and $a(125) \approx 64.54$. The maximum acceleration is about 64.5 ft/s^2 and the minimum acceleration is about 21.93 ft/s^2.

73. (a) $v(r) = k(r_0 - r)r^2 = kr_0r^2 - kr^3 \Rightarrow v'(r) = 2kr_0r - 3kr^2$. $v'(r) = 0 \Rightarrow kr(2r_0 - 3r) = 0$ $\Rightarrow r = 0$ or $\frac{2}{3}r_0$ (but 0 is not in the interval). Evaluating v at $\frac{1}{2}r_0$, $\frac{2}{3}r_0$, and r_0, we get $v(\frac{1}{2}r_0) = \frac{1}{8}kr_0^3$, $v(\frac{2}{3}r_0) = \frac{4}{27}kr_0^3$, and $v(r_0) = 0$. Since $\frac{4}{27} > \frac{1}{8}$, v attains its maximum value at $r = \frac{2}{3}r_0$. This supports the statement in the text.

(b) From part (a), the maximum value of v is $\frac{4}{27}kr_0^3$.

(c)



74. $g(x) = 2 + (x-5)^3 \Rightarrow g'(x) = 3(x-5)^2 \Rightarrow g'(5) = 0$, so 5 is a critical number. But $g(5) = 2$ and g takes on values > 2 and values < 2 in any open interval containing 5, so g does not have a local maximum or minimum at 5.

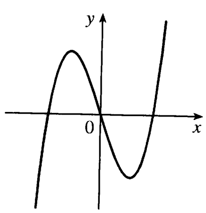
75. $f(x) = x^{101} + x^{51} + x + 1 \Rightarrow f'(x) = 101x^{100} + 51x^{50} + 1 \geq 1$ for all x, so $f'(x) = 0$ has no solution. Thus, $f(x)$ has no critical number, so $f(x)$ can have no local maximum or minimum.

76. Suppose that f has a minimum value at c, so $f(x) \geq f(c)$ for all x near c. Then $g(x) = -f(x) \leq -f(c) = g(c)$ for all x near c, so $g(x)$ has a maximum value at c.

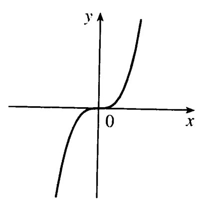
77. If f has a local minimum at c, then $g(x) = -f(x)$ has a local maximum at c, so $g'(c) = 0$ by the case of Fermat's Theorem proved in the text. Thus, $f'(c) = -g'(c) = 0$.

78. (a) $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. So $f'(x) = 3ax^2 + 2bx + c$ is a quadratic and hence has either 2, 1, or 0 real roots, so $f(x)$ has either 2, 1 or 0 critical numbers.

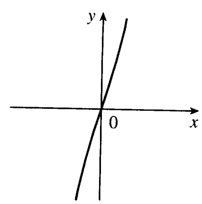
Case (i) (2 critical numbers): $f(x) = x^3 - 3x \Rightarrow$ $f'(x) = 3x^2 - 3$, so $x = -1, 1$ are critical numbers.



Case (ii) (1 critical number): $f(x) = x^3 \Rightarrow$ $f'(x) = 3x^2$, so $x = 0$ is the only critical number.



Case (iii) (no critical number): $f(x) = x^3 + 3x \Rightarrow$ $f'(x) = 3x^2 + 3$, so there are no real roots.



(b) Since there are at most two critical numbers, it can have at most two local extreme values and by (i) this can occur. By (iii) it can have no local extreme value. However, if there is only one critical number, then there is no local extreme value.

APPLIED PROJECT The Calculus of Rainbows

1. From Snell's Law, we have $\sin\alpha = k\sin\beta \approx \frac{4}{3}\sin\beta \Leftrightarrow \beta \approx \arcsin\left(\frac{3}{4}\sin\alpha\right)$. We substitute this into $D(\alpha) = \pi + 2\alpha - 4\beta = \pi + 2\alpha - 4\arcsin\left(\frac{3}{4}\sin\alpha\right)$, and then differentiate to find the minimum: $D'(\alpha) = 2 - 4\left[1 - \left(\frac{3}{4}\sin\alpha\right)^2\right]^{-1/2}\left(\frac{3}{4}\cos\alpha\right) = 2 - \dfrac{3\cos\alpha}{\sqrt{1 - \frac{9}{16}\sin^2\alpha}}$. This is 0 when $\dfrac{3\cos\alpha}{\sqrt{1 - \frac{9}{16}\sin^2\alpha}} = 2$ $\Leftrightarrow \frac{9}{4}\cos^2\alpha = 1 - \frac{9}{16}\sin^2\alpha \Leftrightarrow \frac{9}{4}\cos^2\alpha = 1 - \frac{9}{16}\left(1 - \cos^2\alpha\right) \Leftrightarrow \frac{27}{16}\cos^2\alpha = \frac{7}{16} \Leftrightarrow$ $\cos\alpha = \sqrt{\frac{7}{27}} \Leftrightarrow \alpha = \arccos\sqrt{\frac{7}{27}} \approx 59.4^\circ$, and so the local minimum is $D(59.4^\circ) \approx 2.4$ radians $\approx 138^\circ$. To see that this is an absolute minimum, we check the endpoints, which in this case are $\alpha = 0$ and $\alpha = \frac{\pi}{2}$: $D(0) = \pi$ radians $= 180^\circ$, and $D\left(\frac{\pi}{2}\right) \approx 166^\circ$.

Another method: We first calculate $\dfrac{d\beta}{d\alpha}$: $\sin\alpha = \frac{4}{3}\sin\beta \Leftrightarrow \cos\alpha = \frac{4}{3}\cos\beta\,\dfrac{d\beta}{d\alpha} \Leftrightarrow \dfrac{d\beta}{d\alpha} = \dfrac{3\cos\alpha}{4\cos\beta}$, so since $D'(\alpha) = 2 - 4\dfrac{d\beta}{d\alpha} = 0 \Leftrightarrow \dfrac{d\beta}{d\alpha} = \dfrac{1}{2}$, the minimum occurs when $3\cos\alpha = 2\cos\beta$. Now we square both sides and substitute $\sin\alpha = \frac{4}{3}\sin\beta$, leading to the same result.

2. If we repeat Problem 1 with k in place of $\frac{4}{3}$, we get $D(\alpha) = \pi + 2\alpha - 4\arcsin\left(\dfrac{1}{k}\sin\alpha\right) \Rightarrow$

$D'(\alpha) = 2 - \dfrac{4\cos\alpha}{k\sqrt{1 - [(\sin\alpha)/k]^2}}$, which is 0 when $\dfrac{2\cos\alpha}{k} = \sqrt{1 - \left(\dfrac{\sin\alpha}{k}\right)^2} \Leftrightarrow$

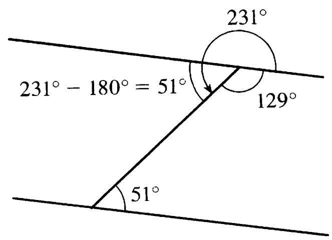
$\left(\dfrac{2\cos\alpha}{k}\right)^2 = 1 - \left(\dfrac{\sin\alpha}{k}\right)^2 \Leftrightarrow 4\cos^2\alpha = k^2 - \sin^2\alpha \Leftrightarrow 3\cos^2\alpha = k^2 - 1 \Leftrightarrow$

$\alpha = \arccos\sqrt{\dfrac{k^2-1}{3}}$. So for $k \approx 1.3318$ (red light) the minimum occurs at $\alpha_1 \approx 1.038$ radians, and so the rainbow angle is about $\pi - D(\alpha_1) \approx 42.3^\circ$. For $k \approx 1.3435$ (violet light) the minimum occurs at $\alpha_2 \approx 1.026$ radians, and so the rainbow angle is about $\pi - D(\alpha_2) \approx 40.6^\circ$.

Another method: As in Problem 1, we can instead find $D'(\alpha)$ in terms of $\dfrac{d\beta}{d\alpha}$, and then substitute $\dfrac{d\beta}{d\alpha} = \dfrac{\cos\alpha}{k\cos\beta}$.

3. At each reflection or refraction, the light is bent in a counterclockwise direction: the bend at A is $\alpha - \beta$, the bend at B is $\pi - 2\beta$, the bend at C is again $\pi - 2\beta$, and the bend at D is $\alpha - \beta$. So the total bend is $D(\alpha) = 2(\alpha - \beta) + 2(\pi - 2\beta) = 2\alpha - 6\beta + 2\pi$, as required. We substitute $\beta = \arcsin\left(\dfrac{\sin\alpha}{k}\right)$ and differentiate, to get $D'(\alpha) = 2 - \dfrac{6\cos\alpha}{k\sqrt{1 - [(\sin\alpha)/k]^2}}$, which is 0 when $\dfrac{3\cos\alpha}{k} = \sqrt{1 - \left(\dfrac{\sin\alpha}{k}\right)^2} \Leftrightarrow$

$9\cos^2\alpha = k^2 - \sin^2\alpha \Leftrightarrow 8\cos^2\alpha = k^2 - 1 \Leftrightarrow \cos\alpha = \sqrt{\frac{1}{8}(k^2 - 1)}$. If $k = \frac{4}{3}$, then the minimum occurs at $\alpha_1 = \arccos\sqrt{\dfrac{(4/3)^2 - 1}{8}} \approx 1.254$ radians. Thus, the minimum *counterclockwise* rotation is $D(\alpha_1) \approx 231^\circ$, which is equivalent to a *clockwise* rotation of $360^\circ - 231^\circ = 129^\circ$ (see the figure). So the rainbow angle for the secondary rainbow is about $180^\circ - 129^\circ = 51^\circ$, as required. In general, the rainbow angle for the secondary rainbow is $\pi - [2\pi - D(\alpha)] = D(\alpha) - \pi$.



4. In the primary rainbow, the rainbow angle gets smaller as k gets larger, as we found in Problem 2, so the colors appear from top to bottom in order of increasing k. But in the secondary rainbow, the rainbow angle gets larger as k gets larger. To see this, we find the minimum deviations for red light and for violet light in the secondary rainbow.

For $k \approx 1.3318$ (red light) the minimum occurs at $\alpha_1 \approx \arccos\sqrt{\dfrac{1.3318^2 - 1}{8}} \approx 1.255$ radians, and so the rainbow angle is $D(\alpha_1) - \pi \approx 50.6^\circ$. For $k \approx 1.3435$ (violet light) the minimum occurs at $\alpha_2 \approx \arccos\sqrt{\dfrac{1.3435^2 - 1}{8}} \approx 1.248$ radians, and so the rainbow angle is $D(\alpha_2) - \pi \approx 53.6^\circ$. Consequently, the rainbow angle is larger for colors with higher indices of refraction, and the colors appear from bottom to top in order of increasing k, the reverse of their order in the primary rainbow.

Note that our calculations above also explain why the secondary rainbow is more spread out than the primary rainbow: in the primary rainbow, the difference between rainbow angles for red and violet light is about 1.7°, whereas in the secondary rainbow it is about 3°.

4.2 The Mean Value Theorem

1. $f(x) = x^2 - 4x + 1$, $[0, 4]$. Since f is a polynomial, it is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[0, 4]$ and differentiable on $(0, 4)$. Also, $f(0) = 1 = f(4)$. $f'(c) = 0 \;\Leftrightarrow\; 2c - 4 = 0 \;\Leftrightarrow\; c = 2$, which is in the open interval $(0, 4)$, so $c = 2$ satisfies the conclusion of Rolle's Theorem.

2. $f(x) = x^3 - 3x^2 + 2x + 5$, $[0, 2]$. f is continuous on $[0, 2]$ and differentiable on $(0, 2)$. Also, $f(0) = 5 = f(2)$.

$f'(c) = 0 \;\Leftrightarrow\; 3c^2 - 6c + 2 = 0 \;\Leftrightarrow\; c = \dfrac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \frac{1}{3}\sqrt{3}$, both in $(0, 2)$.

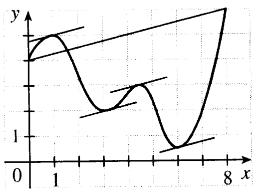
3. $f(x) = \sin 2\pi x$, $[-1, 1]$. f, being the composite of the sine function and the polynomial $2\pi x$, is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Also, $f(-1) = 0 = f(1)$. $f'(c) = 0 \;\Leftrightarrow\; 2\pi\cos 2\pi c = 0 \;\Leftrightarrow\; \cos 2\pi c = 0 \;\Leftrightarrow\; 2\pi c = \pm\frac{\pi}{2} + 2\pi n \;\Leftrightarrow\; c = \pm\frac{1}{4} + n$. If $n = 0$ or ± 1, then $c = \pm\frac{1}{4}, \pm\frac{3}{4}$ is in $(-1, 1)$.

4. $f(x) = x\sqrt{x + 6}$, $[-6, 0]$. f is continuous on its domain, $[-6, \infty)$, and differentiable on $(-6, \infty)$, so it is continuous on $[-6, 0]$ and differentiable on $(-6, 0)$. Also, $f(-6) = 0 = f(0)$. $f'(c) = 0 \;\Leftrightarrow\; \dfrac{3c + 12}{2\sqrt{c + 6}} = 0 \;\Leftrightarrow\; c = -4$, which is in $(-6, 0)$.

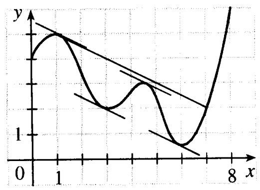
5. $f(x) = 1 - x^{2/3}$. $f(-1) = 1 - (-1)^{2/3} = 1 - 1 = 0 = f(1)$. $f'(x) = -\frac{2}{3}x^{-1/3}$, so $f'(c) = 0$ has no solution. This does not contradict Rolle's Theorem, since $f'(0)$ does not exist, and so f is not differentiable on $(-1, 1)$.

6. $f(x) = (x - 1)^{-2}$. $f(0) = (0 - 1)^{-2} = 1 = (2 - 1)^{-2} = f(2)$. $f'(x) = -2(x - 1)^{-3} \;\Rightarrow\; f'(x)$ is never 0. This does not contradict Rolle's Theorem since $f'(1)$ does not exist.

7. $\dfrac{f(8)-f(0)}{8-0}=\dfrac{6-4}{8}=\dfrac{1}{4}$. The values of c which satisfy $f'(c)=\frac{1}{4}$ seem to be about $c=0.8, 3.2, 4.4$, and 6.1.

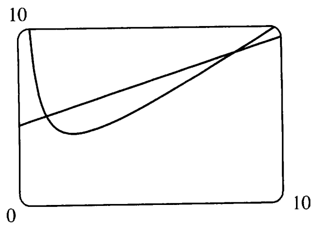


8. $\dfrac{f(7)-f(1)}{7-1}=\dfrac{2-5}{6}=-\dfrac{1}{2}$. The values of c which satisfy $f'(c)=-\frac{1}{2}$ seem to be about $c=1.1, 2.8, 4.6$, and 5.8.

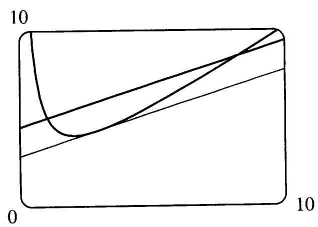


9. (a), (b) The equation of the secant line is

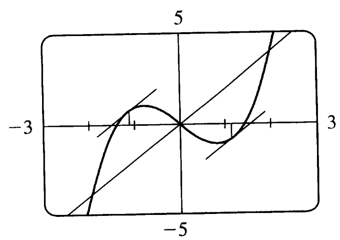
$$y-5=\frac{8.5-5}{8-1}(x-1) \quad\Leftrightarrow$$
$$y=\tfrac{1}{2}x+\tfrac{9}{2}.$$



(c) $f(x)=x+4/x \;\Rightarrow\; f'(x)=1-4/x^2$.
So $f'(c)=\frac{1}{2} \;\Rightarrow\; c^2=8 \;\Rightarrow\; c=2\sqrt{2}$, and $f(c)=2\sqrt{2}+\frac{4}{2\sqrt{2}}=3\sqrt{2}$. Thus, an equation of the tangent line is $y-3\sqrt{2}=\frac{1}{2}(x-2\sqrt{2}) \;\Leftrightarrow\; y=\frac{1}{2}x+2\sqrt{2}$.



10. (a)



It seems that the tangent lines are parallel to the secant at $x\approx\pm1.2$.

(b) The slope of the secant line is 2, and its equation is $y=2x$. $f(x)=x^3-2x \;\Rightarrow\; f'(x)=3x^2-2$, so we solve $f'(c)=2 \;\Rightarrow\; 3c^2=4 \;\Rightarrow\; c=\pm\frac{2\sqrt{3}}{3}\approx1.155$. Our estimates were off by about 0.045 in each case.

11. $f(x)=3x^2+2x+5$, $[-1,1]$. f is continuous on $[-1,1]$ and differentiable on $(-1,1)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c)=\dfrac{f(b)-f(a)}{b-a} \;\Leftrightarrow\; 6c+2=\dfrac{f(1)-f(-1)}{1-(-1)}=\dfrac{10-6}{2}=2 \;\Leftrightarrow\; 6c=0 \;\Leftrightarrow\; c=0$, which is in $(-1,1)$.

12. $f(x) = x^3 + x - 1$, $[0, 2]$. f is continuous on $[0, 2]$ and differentiable on $(0, 2)$. $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} \iff$ $3c^2 + 1 = \dfrac{9 - (-1)}{2} \iff 3c^2 = 5 - 1 \iff c^2 = \frac{4}{3} \iff c = \pm\frac{2}{\sqrt{3}}$, but only $\frac{2}{\sqrt{3}}$ is in $(0, 2)$.

13. $f(x) = e^{-2x}$, $[0, 3]$. f is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[0, 3]$ and differentiable on $(0, 3)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \iff -2e^{-2c} = \dfrac{e^{-6} - e^0}{3 - 0} \iff e^{-2c} = \dfrac{1 - e^{-6}}{6} \iff -2c = \ln\left(\dfrac{1 - e^{-6}}{6}\right)$ $\iff c = -\dfrac{1}{2}\ln\left(\dfrac{1 - e^{-6}}{6}\right) \approx 0.897$, which is in $(0, 3)$.

14. $f(x) = \dfrac{x}{x + 2}$, $[1, 4]$. f is continuous on $[1, 4]$ and differentiable on $(1, 4)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \iff$ $\dfrac{2}{(c + 2)^2} = \dfrac{\frac{2}{3} - \frac{1}{3}}{4 - 1} \iff (c + 2)^2 = 18 \iff c = -2 \pm 3\sqrt{2}$. $-2 + 3\sqrt{2} \approx 2.24$ is in $(1, 4)$.

15. $f(x) = |x - 1|$. $f(3) - f(0) = |3 - 1| - |0 - 1| = 1$. Since $f'(c) = -1$ if $c < 1$ and $f'(c) = 1$ if $c > 1$, $f'(c)(3 - 0) = \pm 3$ and so is never equal to 1. This does not contradict the Mean Value Theorem since $f'(1)$ does not exist.

16. $f(x) = \dfrac{x + 1}{x - 1}$. $f(2) - f(0) = 3 - (-1) = 4$. $f'(x) = \dfrac{1(x - 1) - 1(x + 1)}{(x - 1)^2} = \dfrac{-2}{(x - 1)^2}$. Since $f'(x) < 0$ for all x (except $x = 1$), $f'(c)(2 - 0)$ is always < 0 and hence cannot equal 4. This does not contradict the Mean Value Theorem since f is not continuous at $x = 1$.

17. Let $f(x) = 1 + 2x + x^3 + 4x^5$. Then $f(-1) = -6 < 0$ and $f(0) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem says that there is a number c between -1 and 0 such that $f(c) = 0$. Thus, the given equation has a real root. Suppose the equation has distinct real roots a and b with $a < b$. Then $f(a) = f(b) = 0$. Since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$. By Rolle's Theorem, there is a number r in (a, b) such that $f'(r) = 0$. But $f'(x) = 2 + 3x^2 + 20x^4 \geq 2$ for all x, so $f'(x)$ can never be 0. This contradiction shows that the equation can't have two distinct real roots. Hence, it has exactly one real root.

18. Let $f(x) = 2x - 1 - \sin x$. Then $f(0) = -1 < 0$ and $f(\pi/2) = \pi - 2 > 0$. f is the sum of the polynomial $2x - 1$ and the scalar multiple $(-1) \cdot \sin x$ of the trigonometric function $\sin x$, so f is continuous (and differentiable) for all x. By the Intermediate Value Theorem, there is a number c in $(0, \pi/2)$ such that $f(c) = 0$. Thus, the given equation has at least one real root. If the equation has distinct real roots a and b with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. But $f'(r) = 2 - \cos r > 0$ since $\cos r \leq 1$. This contradiction shows that the given equation can't have two distinct real roots, so it has exactly one real root.

19. Let $f(x) = x^3 - 15x + c$ for x in $[-2, 2]$. If f has two real roots a and b in $[-2, 2]$, with $a < b$, then $f(a) = f(b) = 0$. Since the polynomial f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. Now $f'(r) = 3r^2 - 15$. Since r is in (a, b), which is contained in $[-2, 2]$, we have $|r| < 2$, so $r^2 < 4$. It follows that $3r^2 - 15 < 3 \cdot 4 - 15 = -3 < 0$. This contradicts $f'(r) = 0$, so the given equation can't have two real roots in $[-2, 2]$. Hence, it has at most one real root in $[-2, 2]$.

20. $f(x) = x^4 + 4x + c$. Suppose that $f(x) = 0$ has three distinct real roots a, b, d where $a < b < d$. Then $f(a) = f(b) = f(d) = 0$. By Rolle's Theorem there are numbers c_1 and c_2 with $a < c_1 < b$ and $b < c_2 < d$ and $0 = f'(c_1) = f'(c_2)$, so $f'(x) = 0$ must have at least two real solutions. However $0 = f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 4(x + 1)(x^2 - x + 1)$ has as its only real solution $x = -1$. Thus, $f(x)$ can have at most two real roots.

21. (a) Suppose that a cubic polynomial $P(x)$ has roots $a_1 < a_2 < a_3 < a_4$, so $P(a_1) = P(a_2) = P(a_3) = P(a_4)$. By Rolle's Theorem there are numbers c_1, c_2, c_3 with $a_1 < c_1 < a_2$, $a_2 < c_2 < a_3$ and $a_3 < c_3 < a_4$ and $P'(c_1) = P'(c_2) = P'(c_3) = 0$. Thus, the second-degree polynomial $P'(x)$ has three distinct real roots, which is impossible.

(b) We prove by induction that a polynomial of degree n has at most n real roots. This is certainly true for $n = 1$. Suppose that the result is true for all polynomials of degree n and let $P(x)$ be a polynomial of degree $n + 1$. Suppose that $P(x)$ has more than $n + 1$ real roots, say $a_1 < a_2 < a_3 < \cdots < a_{n+1} < a_{n+2}$. Then $P(a_1) = P(a_2) = \cdots = P(a_{n+2}) = 0$. By Rolle's Theorem there are real numbers $c_1, \ldots, c_{n+1}$ with $a_1 < c_1 < a_2, \ldots, a_{n+1} < c_{n+1} < a_{n+2}$ and $P'(c_1) = \cdots = P'(c_{n+1}) = 0$. Thus, the nth degree polynomial $P'(x)$ has at least $n + 1$ roots. This contradiction shows that $P(x)$ has at most $n + 1$ real roots.

22. (a) Suppose that $f(a) = f(b) = 0$ where $a < b$. By Rolle's Theorem applied to f on $[a, b]$ there is a number c such that $a < c < b$ and $f'(c) = 0$.

(b) Suppose that $f(a) = f(b) = f(c) = 0$ where $a < b < c$. By Rolle's Theorem applied to $f(x)$ on $[a, b]$ and $[b, c]$ there are numbers $a < d < b$ and $b < e < c$ with $f'(d) = 0$ and $f'(e) = 0$. By Rolle's Theorem applied to $f'(x)$ on $[d, e]$ there is a number g with $d < g < e$ such that $f''(g) = 0$.

(c) Suppose that f is n times differentiable on $\mathbb{R}$ and has $n + 1$ distinct real roots. Then $f^{(n)}$ has at least one real root.

23. By the Mean Value Theorem, $f(4) - f(1) = f'(c)(4 - 1)$ for some $c \in (1, 4)$. But for every $c \in (1, 4)$ we have $f'(c) \geq 2$. Putting $f'(c) \geq 2$ into the above equation and substituting $f(1) = 10$, we get $f(4) = f(1) + f'(c)(4 - 1) = 10 + 3f'(c) \geq 10 + 3 \cdot 2 = 16$. So the smallest possible value of $f(4)$ is 16.

24. If $3 \leq f'(x) \leq 5$ for all x, then by the Mean Value Theorem, $f(8) - f(2) = f'(c) \cdot (8 - 2)$ for some c in $[2, 8]$. (f is differentiable for all x, so, in particular, f is differentiable on $(2, 8)$ and continuous on $[2, 8]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) Since $f(8) - f(2) = 6f'(c)$ and $3 \leq f'(c) \leq 5$, it follows that $6 \cdot 3 \leq 6f'(c) \leq 6 \cdot 5 \quad \Rightarrow \quad 18 \leq f(8) - f(2) \leq 30$.

25. Suppose that such a function f exists. By the Mean Value Theorem there is a number $0 < c < 2$ with $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{5}{2}$. But this is impossible since $f'(x) \leq 2 < \frac{5}{2}$ for all x, so no such function can exist.

26. Let $h = f - g$. Then since f and g are continuous on $[a, b]$ and differentiable on (a, b), so is h, and thus h satisfies the assumptions of the Mean Value Theorem. Therefore, there is a number c with $a < c < b$ such that $h(b) = h(b) - h(a) = h'(c)(b - a)$. Since $h'(c) < 0$, $h'(c)(b - a) < 0$, so $f(b) - g(b) = h(b) < 0$ and hence $f(b) < g(b)$.

27. We use Exercise 26 with $f(x) = \sqrt{1+x}$, $g(x) = 1 + \frac{1}{2}x$, and $a = 0$. Notice that $f(0) = 1 = g(0)$ and $f'(x) = \dfrac{1}{2\sqrt{1+x}} < \dfrac{1}{2} = g'(x)$ for $x > 0$. So by Exercise 26, $f(b) < g(b) \quad \Rightarrow \quad \sqrt{1+b} < 1 + \frac{1}{2}b$ for $b > 0$.

Another method: Apply the Mean Value Theorem directly to either $f(x) = 1 + \frac{1}{2}x - \sqrt{1+x}$ or $g(x) = \sqrt{1+x}$ on $[0, b]$.

28. f satisfies the conditions for the Mean Value Theorem, so we use this theorem on the interval $[-b, b]$: $\dfrac{f(b) - f(-b)}{b - (-b)} = f'(c)$ for some $c \in (-b, b)$. But since f is odd, $f(-b) = -f(b)$. Substituting this into the above equation, we get $\dfrac{f(b) + f(b)}{2b} = f'(c) \quad \Rightarrow \quad \dfrac{f(b)}{b} = f'(c)$.

29. Let $f(x) = \sin x$ and let $b < a$. Then $f(x)$ is continuous on $[b, a]$ and differentiable on (b, a). By the Mean Value Theorem, there is a number $c \in (b, a)$ with $\sin a - \sin b = f(a) - f(b) = f'(c)(a - b) = (\cos c)(a - b)$. Thus, $|\sin a - \sin b| \le |\cos c|\,|b - a| \le |a - b|$. If $a < b$, then $|\sin a - \sin b| = |\sin b - \sin a| \le |b - a| = |a - b|$. If $a = b$, both sides of the inequality are 0.

30. Suppose that $f'(x) = c$. Let $g(x) = cx$, so $g'(x) = c$. Then, by Corollary 7, $f(x) = g(x) + d$, where d is a constant, so $f(x) = cx + d$.

31. For $x > 0$, $f(x) = g(x)$, so $f'(x) = g'(x)$. For $x < 0$, $f'(x) = (1/x)' = -1/x^2$ and $g'(x) = (1 + 1/x)' = -1/x^2$, so again $f'(x) = g'(x)$. However, the domain of $g(x)$ is not an interval [it is $(-\infty, 0) \cup (0, \infty)$] so we cannot conclude that $f - g$ is constant (in fact it is not).

32. Let $f(x) = 2\sin^{-1} x - \cos^{-1}(1 - 2x^2)$. Then

$$f'(x) = \frac{2}{\sqrt{1 - x^2}} - \frac{4x}{\sqrt{1 - (1 - 2x^2)^2}} = \frac{2}{\sqrt{1 - x^2}} - \frac{4x}{2x\sqrt{1 - x^2}} = 0$$ (since $x \ge 0$). Thus, $f'(x) = 0$ for all $x \in (0, 1)$. Thus, $f(x) = C$ on $(0, 1)$. To find C, let $x = 0.5$. Thus, $2\sin^{-1}(0.5) - \cos^{-1}(0.5) = 2\left(\frac{\pi}{6}\right) - \frac{\pi}{3} = 0 = C$. We conclude that $f(x) = 0$ for x in $(0, 1)$. By continuity of f, $f(x) = 0$ on $[0, 1]$. Therefore, we see that $f(x) = 2\,\sin^{-1} x - \cos^{-1}(1 - 2x^2) = 0 \quad \Rightarrow$ $2\,\sin^{-1} x = \cos^{-1}(1 - 2x^2)$.

33. Let $f(x) = \arcsin\left(\dfrac{x - 1}{x + 1}\right) - 2\,\arctan\sqrt{x} + \frac{\pi}{2}$. Note that the domain of f is $[0, \infty)$. Thus,

$$f'(x) = \frac{1}{\sqrt{1 - \left(\dfrac{x - 1}{x + 1}\right)^2}}\,\frac{(x + 1) - (x - 1)}{(x + 1)^2} - \frac{2}{1 + x} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}\,(x + 1)} - \frac{1}{\sqrt{x}\,(x + 1)} = 0.$$ Then $f(x) = C$ on $(0, \infty)$ by Theorem 5. By continuity of f, $f(x) = C$ on $[0, \infty)$. To find C, we let $x = 0 \quad \Rightarrow$ $\arcsin(-1) - 2\arctan(0) + \frac{\pi}{2} = C \quad \Rightarrow \quad -\frac{\pi}{2} - 0 + \frac{\pi}{2} = 0 = C$. Thus, $f(x) = 0 \quad \Rightarrow$ $\arcsin\left(\dfrac{x - 1}{x + 1}\right) = 2\arctan\sqrt{x} - \frac{\pi}{2}$.

34. Let $v(t)$ be the velocity of the car t hours after 2:00 P.M. Then $\dfrac{v(1/6)-v(0)}{1/6-0}=\dfrac{50-30}{1/6}=120$. By the Mean Value Theorem, there is a number c such that $0<c<\frac{1}{6}$ with $v'(c)=120$. Since $v'(t)$ is the acceleration at time t, the acceleration c hours after 2:00 P.M. is exactly $120\ \mathrm{mi/h^2}$.

35. Let $g(t)$ and $h(t)$ be the position functions of the two runners and let $f(t)=g(t)-h(t)$. By hypothesis, $f(0)=g(0)-h(0)=0$ and $f(b)=g(b)-h(b)=0$, where b is the finishing time. Then by the Mean Value Theorem, there is a time c, with $0<c<b$, such that $f'(c)=\dfrac{f(b)-f(0)}{b-0}$. But $f(b)=f(0)=0$, so $f'(c)=0$. Since $f'(c)=g'(c)-h'(c)=0$, we have $g'(c)=h'(c)$. So at time c, both runners have the same speed $g'(c)=h'(c)$.

36. Assume that f is differentiable (and hence continuous) on $\mathbb{R}$ and that $f'(x)\neq 1$ for all x. Suppose f has more than one fixed point. Then there are numbers a and b such that $a<b$, $f(a)=a$, and $f(b)=b$. Applying the Mean Value Theorem to the function f on $[a,b]$, we find that there is a number c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

But then $f'(c)=\dfrac{b-a}{b-a}=1$, contradicting our assumption that $f'(x)\neq 1$ for every real number x. This shows that our supposition was wrong, that is, that f cannot have more than one fixed point.

4.3 How Derivatives Affect the Shape of a Graph

1. (a) f is increasing on $(0,6)$ and $(8,9)$.

(b) f is decreasing on $(6,8)$.

(c) f is concave upward on $(2,4)$ and $(7,9)$.

(d) f is concave downward on $(0,2)$ and $(4,7)$.

(e) The points of inflection are $(2,3)$, $(4,4.5)$ and $(7,4)$ (where the concavity changes).

2. (a) f is increasing on $(1,\approx 3.8)$ and $(5,\approx 6.5)$.

(b) f is decreasing on $(0,1)$, $(\approx 3.8,5)$, $(\approx 6.5,8)$, and $(8,9)$.

(c) f is concave upward on $(0,3)$ and $(8,9)$.

(d) f is concave downward on $(3,5)$ and $(5,8)$.

(e) The point of inflection is $(3,\approx 1.8)$ (where the concavity changes).

3. (a) Use the Increasing/Decreasing (I/D) Test.

(b) Use the Concavity Test.

(c) At any value of x where the concavity changes, we have an inflection point at $(x,f(x))$.

4. (a) See the First Derivative Test.

(b) See the Second Derivative Test and the note that precedes Example 7.

5. (a) Since $f'(x)>0$ on $(1,5)$, f is increasing on this interval. Since $f'(x)<0$ on $(0,1)$ and $(5,6)$, f is decreasing on these intervals.

(b) Since $f'(x)=0$ at $x=1$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x=1$. Since $f'(x)=0$ at $x=5$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x=5$.

6. (a) $f'(x) > 0$ and f is increasing on $(0, 1)$ and $(3, 5)$. $f'(x) < 0$ and f is decreasing on $(1, 3)$ and $(5, 6)$.

(b) Since $f'(x) = 0$ at $x = 1$ and $x = 5$ and f' changes from positive to negative at both values, f changes from increasing to decreasing and has local maxima at $x = 1$ and $x = 5$. Since $f'(x) = 0$ at $x = 3$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x = 3$.

7. There is an inflection point at $x = 1$ because $f''(x)$ changes from negative to positive there, and so the graph of f changes from concave downward to concave upward. There is an inflection point at $x = 7$ because $f''(x)$ changes from positive to negative there, and so the graph of f changes from concave upward to concave downward.

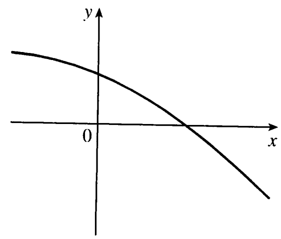
8. (a) f is increasing on the intervals where $f'(x) > 0$, namely, $(2, 4)$ and $(6, 9)$.

(b) f has a local maximum where it changes from increasing to decreasing, that is, where f' changes from positive to negative (at $x = 4$). Similarly, where f' changes from negative to positive, f has a local minimum (at $x = 2$ and at $x = 6$).

(c) When f' is increasing, its derivative f'' is positive and hence, f is concave upward. This happens on $(1, 3)$, $(5, 7)$, and $(8, 9)$. Similarly, f is concave downward when f' is decreasing—that is, on $(0, 1)$, $(3, 5)$, and $(7, 8)$.

(d) f has inflection points at $x = 1, 3, 5, 7$, and 8, since the direction of concavity changes at each of these values.

9. The function must be always decreasing and concave downward.



10. (a) The rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t = 8$ hours, and decreases toward 0 as the population begins to level off.

(b) The rate of increase has its maximum value at $t = 8$ hours.

(c) The population function is concave upward on $(0, 8)$ and concave downward on $(8, 18)$.

(d) At $t = 8$, the population is about 350, so the inflection point is about $(8, 350)$.

11. (a) $f(x) = x^3 - 12x + 1 \quad \Rightarrow \quad f'(x) = 3x^2 - 12 = 3(x + 2)(x - 2)$.
We don't need to include "3" in the chart to determine the sign of $f'(x)$.

Interval	$x + 2$	$x - 2$	$f'(x)$	f
$x < -2$	$-$	$-$	$+$	increasing on $(-\infty, -2)$
$-2 < x < 2$	$+$	$-$	$-$	decreasing on $(-2, 2)$
$x > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

So f is increasing on $(-\infty, -2)$ and $(2, \infty)$ and f is decreasing on $(-2, 2)$.

(b) f changes from increasing to decreasing at $x = -2$ and from decreasing to increasing at $x = 2$. Thus, $f(-2) = 17$ is a local maximum value and $f(2) = -15$ is a local minimum value.

(c) $f''(x) = 6x$. $f''(x) > 0 \quad \Leftrightarrow \quad x > 0$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < 0$. Thus, f is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$. There is an inflection point where the concavity changes, at $(0, f(0)) = (0, 1)$.

12. (a) $f(x) = 5 - 3x^2 + x^3 \quad \Rightarrow \quad f'(x) = -6x + 3x^2 = 3x(x-2)$. Thus, $f'(x) > 0 \quad \Leftrightarrow \quad x < 0$ or $x > 2$ and $f'(x) < 0 \quad \Leftrightarrow \quad 0 < x < 2$. So f is increasing on $(-\infty, 0)$ and $(2, \infty)$ and f is decreasing on $(0, 2)$.

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = 2$. Thus, $f(0) = 5$ is a local maximum value and $f(2) = 1$ is a local minimum value.

(c) $f''(x) = -6 + 6x = 6(x-1)$. $f''(x) > 0 \quad \Leftrightarrow \quad x > 1$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < 1$. Thus, f is concave upward on $(1, \infty)$ and concave downward on $(-\infty, 1)$. There is an inflection point at $(1, 3)$.

13. (a) $f(x) = x^4 - 2x^2 + 3 \quad \Rightarrow \quad f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1)$.

Interval	$x+1$	x	$x-1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$+$	$-$	$-$	$+$	increasing on $(-1, 0)$
$0 < x < 1$	$+$	$+$	$-$	$-$	decreasing on $(0, 1)$
$x > 1$	$+$	$+$	$+$	$+$	increasing on $(1, \infty)$

So f is increasing on $(-1, 0)$ and $(1, \infty)$ and f is decreasing on $(-\infty, -1)$ and $(0, 1)$.

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = -1$ and $x = 1$. Thus, $f(0) = 3$ is a local maximum value and $f(\pm 1) = 2$ are local minimum values.

(c) $f''(x) = 12x^2 - 4 = 12\left(x^2 - \frac{1}{3}\right) = 12\left(x + 1/\sqrt{3}\right)\left(x - 1/\sqrt{3}\right)$. $f''(x) > 0 \quad \Leftrightarrow \quad x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$ and $f''(x) < 0 \quad \Leftrightarrow \quad -1/\sqrt{3} < x < 1/\sqrt{3}$. Thus, f is concave upward on $\left(-\infty, -\sqrt{3}/3\right)$ and $\left(\sqrt{3}/3, \infty\right)$ and concave downward on $\left(-\sqrt{3}/3, \sqrt{3}/3\right)$. There are inflection points at $\left(\pm\sqrt{3}/3, \frac{22}{9}\right)$.

14. (a) $f(x) = \dfrac{x^2}{x^2 + 3} \quad \Rightarrow \quad f'(x) = \dfrac{(x^2+3)(2x) - x^2(2x)}{(x^2+3)^2} = \dfrac{6x}{(x^2+3)^2}$. The denominator is positive so the sign of $f'(x)$ is determined by the sign of x. Thus, $f'(x) > 0 \quad \Leftrightarrow \quad x > 0$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < 0$. So f is increasing on $(0, \infty)$ and f is decreasing on $(-\infty, 0)$.

(b) f changes from decreasing to increasing at $x = 0$. Thus, $f(0) = 0$ is a local minimum value.

(c)
$$f''(x) = \frac{(x^2+3)^2(6) - 6x \cdot 2(x^2+3)(2x)}{[(x^2+3)^2]^2} = \frac{6(x^2+3)\left[x^2 + 3 - 4x^2\right]}{(x^2+3)^4}$$
$$= \frac{6(3 - 3x^2)}{(x^2+3)^3} = \frac{-18(x+1)(x-1)}{(x^2+3)^3}.$$

$f''(x) > 0 \quad \Leftrightarrow \quad -1 < x < 1$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < -1$ or $x > 1$. Thus, f is concave upward on $(-1, 1)$ and concave downward on $(-\infty, -1)$ and $(1, \infty)$. There are inflection points at $\left(\pm 1, \frac{1}{4}\right)$.

15. (a) $f(x) = x - 2\sin x$ on $(0, 3\pi) \quad \Rightarrow \quad f'(x) = 1 - 2\cos x$. $f'(x) > 0 \quad \Leftrightarrow \quad 1 - 2\cos x > 0 \quad \Leftrightarrow \quad \cos x < \frac{1}{2} \quad \Leftrightarrow \quad \frac{\pi}{3} < x < \frac{5\pi}{3}$ or $\frac{7\pi}{3} < x < 3\pi$. $f'(x) < 0 \quad \Leftrightarrow \quad \cos x > \frac{1}{2} \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{3}$ or $\frac{5\pi}{3} < x < \frac{7\pi}{3}$. So f is increasing on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and $\left(\frac{7\pi}{3}, 3\pi\right)$, and f is decreasing on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{7\pi}{3}\right)$.

(b) f changes from increasing to decreasing at $x = \frac{5\pi}{3}$, and from decreasing to increasing at $x = \frac{\pi}{3}$ and at $x = \frac{7\pi}{3}$. Thus, $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3} \approx 6.97$ is a local maximum value and $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3} \approx -0.68$ and $f\left(\frac{7\pi}{3}\right) = \frac{7\pi}{3} - \sqrt{3} \approx 5.60$ are local minimum values.

(c) $f''(x) = 2\sin x > 0 \quad \Leftrightarrow \quad 0 < x < \pi$ and $2\pi < x < 3\pi$. $f''(x) < 0 \quad \Leftrightarrow \quad \pi < x < 2\pi$. Thus, f is concave upward on $(0, \pi)$ and $(2\pi, 3\pi)$, and f is concave downward on $(\pi, 2\pi)$. There are inflection points at (π, π) and $(2\pi, 2\pi)$.

16. (a) $f(x) = \cos^2 x - 2\sin x$, $0 \le x \le 2\pi$. $f'(x) = -2\cos x \sin x - 2\cos x = -2\cos x\,(1 + \sin x)$. Note that $1 + \sin x \ge 0$ [since $\sin x \ge -1$], with equality $\Leftrightarrow$ $\sin x = -1$ $\Leftrightarrow$ $x = 3\pi/2$ [since $0 \le x \le 2\pi$] $\Rightarrow$ $\cos x = 0$. Thus, $f'(x) > 0$ $\Leftrightarrow$ $\cos x < 0$ $\Leftrightarrow$ $\pi/2 < x < 3\pi/2$ and $f'(x) < 0$ $\Leftrightarrow$ $\cos x > 0$ $\Leftrightarrow$ $0 < x < \pi/2$ or $3\pi/2 < x < 2\pi$. Thus, f is increasing on $(\pi/2, 3\pi/2)$ and f is decreasing on $(0, \pi/2)$ and $(3\pi/2, 2\pi)$.

(b) f changes from decreasing to increasing at $x = \pi/2$ and from increasing to decreasing at $x = 3\pi/2$. Thus, $f(\pi/2) = -2$ is a local minimum value and $f(3\pi/2) = 2$ is a local maximum value.

(c) $$\begin{aligned} f''(x) &= 2\sin x\,(1 + \sin x) - 2\cos^2 x = 2\sin x + 2\sin^2 x - 2(1 - \sin^2 x) \\ &= 4\sin^2 x + 2\sin x - 2 = 2(2\sin x - 1)(\sin x + 1) \end{aligned}$$

so $f''(x) > 0$ $\Leftrightarrow$ $\sin x > \frac{1}{2}$ $\Leftrightarrow$ $\frac{\pi}{6} < x < \frac{5\pi}{6}$, and $f''(x) < 0$ $\Leftrightarrow$ $\sin x < \frac{1}{2}$ and $\sin x \ne -1$ $\Leftrightarrow$ $0 < x < \frac{\pi}{6}$ or $\frac{5\pi}{6} < x < \frac{3\pi}{2}$ or $\frac{3\pi}{2} < x < 2\pi$. Thus, f is concave upward on $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ and concave downward on $\left(0, \frac{\pi}{6}\right)$, $\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$. There are inflection points at $\left(\frac{\pi}{6}, -\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6}, -\frac{1}{4}\right)$.

17. (a) $y = f(x) = xe^x$ $\Rightarrow$ $f'(x) = xe^x + e^x = e^x(x + 1)$. So $f'(x) > 0$ $\Leftrightarrow$ $x + 1 > 0$ $\Leftrightarrow$ $x > -1$. Thus, f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

(b) f changes from decreasing to increasing at its only critical number, $x = -1$. Thus, $f(-1) = -e^{-1}$ is a local minimum value.

(c) $f'(x) = e^x(x + 1)$ $\Rightarrow$ $f''(x) = e^x(1) + (x + 1)e^x = e^x(x + 2)$. So $f''(x) > 0$ $\Leftrightarrow$ $x + 2 > 0$ $\Leftrightarrow$ $x > -2$. Thus, f is concave upward on $(-2, \infty)$ and concave downward on $(-\infty, -2)$. Since the concavity changes direction at $x = -2$, the point $\left(-2, -2e^{-2}\right)$ is an inflection point.

18. (a) $y = f(x) = x^2 e^x$ $\Rightarrow$ $f'(x) = x^2 e^x + 2xe^x = x(x + 2)e^x$. So $f'(x) > 0$ $\Leftrightarrow$ $x(x + 2) > 0$ $\Leftrightarrow$ either $x < -2$ or $x > 0$. Therefore f is increasing on $(-\infty, -2)$ and $(0, \infty)$, and decreasing on $(-2, 0)$.

(b) f changes from increasing to decreasing at $x = -2$, so $f(-2) = 4e^{-2}$ is a local maximum value. f changes from decreasing to increasing at $x = 0$, so $f(0) = 0$ is a local minimum value.

(c) $f'(x) = \left(x^2 + 2x\right)e^x$ $\Rightarrow$ $f''(x) = \left(x^2 + 2x\right)e^x + e^x(2x + 2) = e^x\left(x^2 + 4x + 2\right)$. $f''(x) = 0$ $\Leftrightarrow$ $x^2 + 4x + 2 = 0$ $\Leftrightarrow$ $x = -2 \pm \sqrt{2}$. $f''(x) < 0$ $\Leftrightarrow$ $-2 - \sqrt{2} < x < -2 + \sqrt{2}$, so f is concave downward on $\left(-2 - \sqrt{2}, -2 + \sqrt{2}\right)$ and concave upward on $\left(-\infty, -2 - \sqrt{2}\right)$ and $\left(-2 + \sqrt{2}, \infty\right)$. There are inflection points at $\left(-2 - \sqrt{2}, f\left(-2 - \sqrt{2}\right)\right) \approx (-3.41, 0.38)$ and $\left(-2 + \sqrt{2}, f\left(-2 + \sqrt{2}\right)\right) \approx (-0.59, 0.19)$.

19. (a) $y = f(x) = \dfrac{\ln x}{\sqrt{x}}$. (Note that f is only defined for $x > 0$.)

$$f'(x) = \frac{\sqrt{x}\,(1/x) - \ln x\left(\frac{1}{2}x^{-1/2}\right)}{x} = \frac{\dfrac{1}{\sqrt{x}} - \dfrac{\ln x}{2\sqrt{x}}}{x} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} = \frac{2 - \ln x}{2x^{3/2}} > 0 \quad\Leftrightarrow$$

$2 - \ln x > 0$ $\Leftrightarrow$ $\ln x < 2$ $\Leftrightarrow$ $x < e^2$. Therefore f is increasing on $\left(0, e^2\right)$ and decreasing on $\left(e^2, \infty\right)$.

(b) f changes from increasing to decreasing at $x = e^2$, so $f\left(e^2\right) = \dfrac{\ln e^2}{\sqrt{e^2}} = \dfrac{2}{e}$ is a local maximum value.

(c) $f''(x) = \dfrac{2x^{3/2}(-1/x) - (2 - \ln x)(3x^{1/2})}{(2x^{3/2})^2} = \dfrac{-2x^{1/2} + 3x^{1/2}(\ln x - 2)}{4x^3}$

$= \dfrac{x^{1/2}(-2 + 3\ln x - 6)}{4x^3} = \dfrac{3\ln x - 8}{4x^{5/2}}$

$f''(x) = 0 \Leftrightarrow \ln x = \frac{8}{3} \Leftrightarrow x = e^{8/3}$. $f''(x) > 0 \Leftrightarrow x > e^{8/3}$, so f is concave upward on $(e^{8/3}, \infty)$ and concave downward on $(0, e^{8/3})$. There is an inflection point at $\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right) \approx (14.39, 0.70)$.

20. (a) $y = f(x) = x\ln x$. (Note that f is only defined for $x > 0$.)

$f'(x) = x(1/x) + \ln x = 1 + \ln x$. $f'(x) > 0 \Leftrightarrow \ln x + 1 > 0 \Leftrightarrow \ln x > -1 \Leftrightarrow x > e^{-1}$.

Therefore f is increasing on $(1/e, \infty)$ and decreasing on $(0, 1/e)$.

(b) f changes from decreasing to increasing at $x = 1/e$, so $f(1/e) = -1/e$ is a local minimum value.

(c) $f''(x) = 1/x > 0$ for $x > 0$. So f is concave upward on its entire domain, and has no inflection point.

21. $f(x) = x^5 - 5x + 3 \Rightarrow f'(x) = 5x^4 - 5 = 5(x^2+1)(x+1)(x-1)$.

First Derivative Test: $f'(x) < 0 \Rightarrow -1 < x < 1$ and $f'(x) > 0 \Rightarrow x > 1$ or $x < -1$. Since f' changes from positive to negative at $x = -1$, $f(-1) = 7$ is a local maximum value; and since f' changes from negative to positive at $x = 1$, $f(1) = -1$ is a local minimum value.

Second Derivative Test: $f''(x) = 20x^3$. $f'(x) = 0 \Leftrightarrow x = \pm 1$. $f''(-1) = -20 < 0 \Rightarrow f(-1) = 7$ is a local maximum value. $f''(1) = 20 > 0 \Rightarrow f(1) = -1$ is a local minimum value.

Preference: For this function, the two tests are equally easy.

22. $f(x) = \dfrac{x}{x^2+4} \Rightarrow f'(x) = \dfrac{(x^2+4)\cdot 1 - x(2x)}{(x^2+4)^2} = \dfrac{4 - x^2}{(x^2+4)^2} = \dfrac{(2+x)(2-x)}{(x^2+4)^2}$.

First Derivative Test: $f'(x) > 0 \Rightarrow -2 < x < 2$ and $f'(x) < 0 \Rightarrow x > 2$ or $x < -2$. Since f' changes from positive to negative at $x = 2$, $f(2) = \frac{1}{4}$ is a local maximum value; and since f' changes from negative to positive at $x = -2$, $f(-2) = -\frac{1}{4}$ is a local minimum value.

Second Derivative Test:

$$f''(x) = \frac{(x^2+4)^2(-2x) - (4 - x^2)\cdot 2(x^2+4)(2x)}{[(x^2+4)^2]^2}$$

$$= \frac{-2x(x^2+4)\left[(x^2+4) + 2(4 - x^2)\right]}{(x^2+4)^4} = \frac{-2x(12 - x^2)}{(x^2+4)^3}$$

$f'(x) = 0 \Leftrightarrow x = \pm 2$. $f''(-2) = \frac{1}{16} > 0 \Rightarrow f(-2) = -\frac{1}{4}$ is a local minimum value.

$f''(2) = -\frac{1}{16} < 0 \Rightarrow f(2) = \frac{1}{4}$ is a local maximum value.

Preference: Since calculating the second derivative is fairly difficult, the First Derivative Test is easier to use for this function.

23. $f(x) = x + \sqrt{1-x} \;\Rightarrow\; f'(x) = 1 + \frac{1}{2}(1-x)^{-1/2}(-1) = 1 - \dfrac{1}{2\sqrt{1-x}}$. Note that f is defined for $1 - x \geq 0$; that is, for $x \leq 1$. $f'(x) = 0 \;\Rightarrow\; 2\sqrt{1-x} = 1 \;\Rightarrow\; \sqrt{1-x} = \frac{1}{2} \;\Rightarrow\; 1 - x = \frac{1}{4} \;\Rightarrow$ $x = \frac{3}{4}$. f' does not exist at $x = 1$, but we can't have a local maximum or minimum at an endpoint.

First Derivative Test: $f'(x) > 0 \;\Rightarrow\; x < \frac{3}{4}$ and $f'(x) < 0 \;\Rightarrow\; \frac{3}{4} < x < 1$. Since f' changes from positive to negative at $x = \frac{3}{4}$, $f\left(\frac{3}{4}\right) = \frac{5}{4}$ is a local maximum value.

Second Derivative Test: $f''(x) = -\frac{1}{2}\left(-\frac{1}{2}\right)(1-x)^{-3/2}(-1) = -\dfrac{1}{4\left(\sqrt{1-x}\right)^3}$. $f''\left(\frac{3}{4}\right) = -2 < 0 \;\Rightarrow$ $f\left(\frac{3}{4}\right) = \frac{5}{4}$ is a local maximum value.

Preference: The First Derivative Test may be slightly easier to apply in this case.

24. (a) $f(x) = x^4(x-1)^3 \;\Rightarrow$

$f'(x) = x^4 \cdot 3(x-1)^2 + (x-1)^3 \cdot 4x^3 = x^3(x-1)^2\,[3x + 4(x-1)] = x^3(x-1)^2(7x-4)$

The critical numbers are 0, 1, and $\frac{4}{7}$.

(b) $f''(x) = 3x^2(x-1)^2(7x-4) + x^3 \cdot 2(x-1)(7x-4) + x^3(x-1)^2 \cdot 7$

$= x^2(x-1)\,[3(x-1)(7x-4) + 2x(7x-4) + 7x(x-1)]$

Now $f''(0) = f''(1) = 0$, so the Second Derivative Test gives no information for $x = 0$ or $x = 1$.

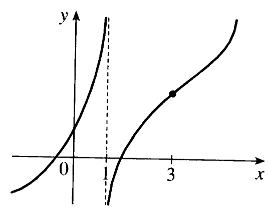
$f''\left(\frac{4}{7}\right) = \left(\frac{4}{7}\right)^2\left(\frac{4}{7} - 1\right)\left[0 + 0 + 7\left(\frac{4}{7}\right)\left(\frac{4}{7} - 1\right)\right] = \left(\frac{4}{7}\right)^2\left(-\frac{3}{7}\right)(4)\left(-\frac{3}{7}\right) > 0$, so there is a local minimum at $x = \frac{4}{7}$.

(c) f' is positive on $(-\infty, 0)$, negative on $\left(0, \frac{4}{7}\right)$, positive on $\left(\frac{4}{7}, 1\right)$, and positive on $(1, \infty)$. So f has a local maximum at $x = 0$, a local minimum at $x = \frac{4}{7}$, and no local maximum or minimum at $x = 1$.

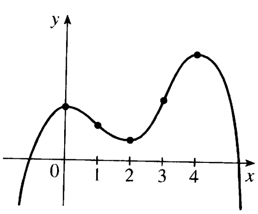
25. (a) By the Second Derivative Test, if $f'(2) = 0$ and $f''(2) = -5 < 0$, f has a local maximum at $x = 2$.

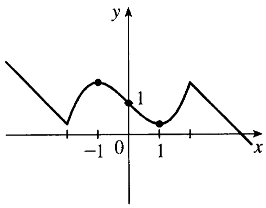
(b) If $f'(6) = 0$, we know that f has a horizontal tangent at $x = 6$. Knowing that $f''(6) = 0$ does not provide any additional information since the Second Derivative Test fails. For example, the first and second derivatives of $y = (x-6)^4$, $y = -(x-6)^4$, and $y = (x-6)^3$ all equal zero for $x = 6$, but the first has a local minimum at $x = 6$, the second has a local maximum at $x = 6$, and the third has an inflection point at $x = 6$.

26. $f'(x) > 0$ for all $x \neq 1$ with vertical asymptote $x = 1$, so f is increasing on $(-\infty, 1)$ and $(1, \infty)$. $f''(x) > 0$ if $x < 1$ or $x > 3$, and $f''(x) < 0$ if $1 < x < 3$, so f is concave upward on $(-\infty, 1)$ and $(3, \infty)$, and concave downward on $(1, 3)$. There is an inflection point when $x = 3$.



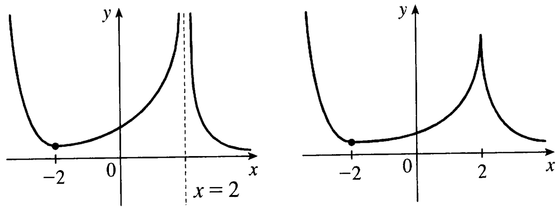
27. $f'(0) = f'(2) = f'(4) = 0 \;\Leftrightarrow\;$ horizontal tangents at $x = 0, 2, 4$. $f'(x) > 0$ if $x < 0$ or $2 < x < 4 \;\Rightarrow\; f$ is increasing on $(-\infty, 0)$ and $(2, 4)$. $f'(x) < 0$ if $0 < x < 2$ or $x > 4 \;\Rightarrow\; f$ is decreasing on $(0, 2)$ and $(4, \infty)$. $f''(x) > 0$ if $1 < x < 3 \;\Rightarrow\; f$ is concave upward on $(1, 3)$. $f''(x) < 0$ if $x < 1$ or $x > 3 \;\Rightarrow\; f$ is concave downward on $(-\infty, 1)$ and $(3, \infty)$. There are inflection points when $x = 1$ and 3.

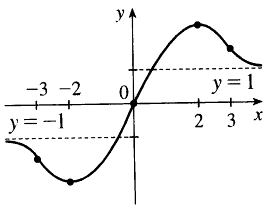


28. 

$f'(1) = f'(-1) = 0 \;\Rightarrow\;$ horizontal tangents at $x = \pm 1$. $f'(x) < 0$ if $|x| < 1 \;\Rightarrow\; f$ is decreasing on $(-1, 1)$. $f'(x) > 0$ if $1 < |x| < 2 \;\Rightarrow\; f$ is increasing on $(-2, -1)$ and $(1, 2)$. $f'(x) = -1$ if $|x| > 2 \;\Rightarrow\;$ the graph of f has constant slope -1 on $(-\infty, -2)$ and $(2, \infty)$. $f''(x) < 0$ if $-2 < x < 0 \;\Rightarrow\; f$ is concave downward on $(-2, 0)$. Inflection point $(0, 1)$.

29. $f'(x) > 0$ if $|x| < 2 \;\Rightarrow\; f$ is increasing on $(-2, 2)$. $f'(x) < 0$ if $|x| > 2 \;\Rightarrow\; f$ is decreasing on $(-\infty, -2)$ and $(2, \infty)$. $f'(-2) = 0 \;\Rightarrow\;$ horizontal tangent at $x = -2$. $\lim\limits_{x \to 2} |f'(x)| = \infty \;\Rightarrow\;$ there is a vertical asymptote or vertical tangent (cusp) at $x = 2$. $f''(x) > 0$ if $x \neq 2 \;\Rightarrow\; f$ is concave upward on $(-\infty, 2)$ and $(2, \infty)$.



30. 

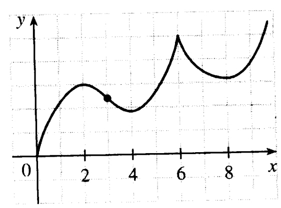
$f'(x) > 0$ if $|x| < 2 \;\Rightarrow\; f$ is increasing on $(-2, 2)$. $f'(x) < 0$ if $|x| > 2 \;\Rightarrow\; f$ is decreasing on $(-\infty, -2)$ and $(2, \infty)$. $f'(2) = 0$, so f has a horizontal tangent (and local maximum) at $x = 2$. $\lim\limits_{x \to \infty} f(x) = 1 \;\Rightarrow\; y = 1$ is a horizontal asymptote. $f(-x) = -f(x) \;\Rightarrow\; f$ is an odd function (its graph is symmetric about the origin). Finally, $f''(x) < 0$ if $0 < x < 3$ and $f''(x) > 0$ if $x > 3$, so f is CD on $(0, 3)$ and CU on $(3, \infty)$.

31. (a) f is increasing where f' is positive, that is, on $(0, 2)$, $(4, 6)$, and $(8, \infty)$; and decreasing where f' is negative, that is, on $(2, 4)$ and $(6, 8)$.

(b) f has local maxima where f' changes from positive to negative, at $x = 2$ and at $x = 6$, and local minima where f' changes from negative to positive, at $x = 4$ and at $x = 8$.

(c) f is concave upward (CU) where f' is increasing, that is, on $(3, 6)$ and $(6, \infty)$, and concave downward (CD) where f' is decreasing, that is, on $(0, 3)$.

(d) There is a point of inflection where f changes from being CD to being CU, that is, at $x = 3$.

(e) 

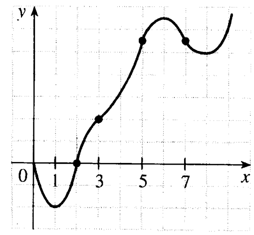
32. (a) f is increasing where f' is positive, on $(1, 6)$ and $(8, \infty)$, and decreasing where f' is negative, on $(0, 1)$ and $(6, 8)$.

(b) f has a local maximum where f' changes from positive to negative, at $x = 6$, and local minima where f' changes from negative to positive, at $x = 1$ and at $x = 8$.

(c) f is concave upward where f' is increasing, that is, on $(0, 2)$, $(3, 5)$, and $(7, \infty)$, and concave downward where f' is decreasing, that is, on $(2, 3)$ and $(5, 7)$.

(d) There are points of inflection where f changes its direction of concavity, at $x = 2$, $x = 3$, $x = 5$ and $x = 7$.

(e)

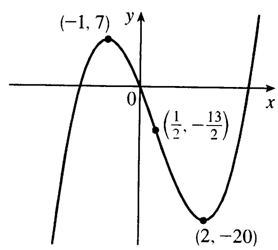


33. (a) $f(x) = 2x^3 - 3x^2 - 12x \Rightarrow f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1)$. $f'(x) > 0 \Leftrightarrow x < -1$ or $x > 2$ and $f'(x) < 0 \Leftrightarrow -1 < x < 2$. So f is increasing on $(-\infty, -1)$ and $(2, \infty)$, and f is decreasing on $(-1, 2)$.

(b) Since f changes from increasing to decreasing at $x = -1$, $f(-1) = 7$ is a local maximum value. Since f changes from decreasing to increasing at $x = 2$, $f(2) = -20$ is a local minimum value.

(c) $f''(x) = 6(2x - 1) \Rightarrow f''(x) > 0$ on $\left(\frac{1}{2}, \infty\right)$ and $f''(x) < 0$ on $\left(-\infty, \frac{1}{2}\right)$. So f is concave upward on $\left(\frac{1}{2}, \infty\right)$ and concave downward on $\left(-\infty, \frac{1}{2}\right)$. There is a change in concavity at $x = \frac{1}{2}$, and we have an inflection point at $\left(\frac{1}{2}, -\frac{13}{2}\right)$.

(d)

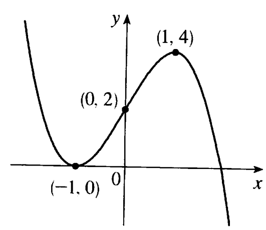


34. (a) $f(x) = 2 + 3x - x^3 \Rightarrow f'(x) = 3 - 3x^2 = -3(x^2 - 1) = -3(x + 1)(x - 1)$. $f'(x) > 0 \Leftrightarrow -1 < x < 1$ and $f'(x) < 0 \Leftrightarrow x < -1$ or $x > 1$. So f is increasing on $(-1, 1)$ and f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) $f(-1) = 0$ is a local minimum value and $f(1) = 4$ is a local maximum value.

(c) $f''(x) = -6x \Rightarrow f''(x) > 0$ on $(-\infty, 0)$ and $f''(x) < 0$ on $(0, \infty)$. So f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$. There is an inflection point at $(0, 2)$.

(d)



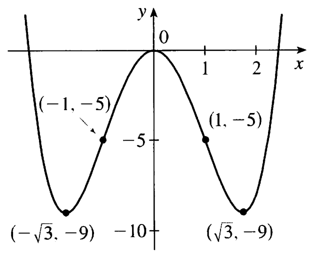
35. (a) $f(x) = x^4 - 6x^2 \Rightarrow f'(x) = 4x^3 - 12x = 4x(x^2 - 3) = 0$ when $x = 0, \pm\sqrt{3}$.

Interval	$4x$	$x^2 - 3$	$f'(x)$	f
$x < -\sqrt{3}$	$-$	$+$	$-$	decreasing on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	$-$	$+$	increasing on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	$-$	$-$	decreasing on $(0, \sqrt{3})$
$x > \sqrt{3}$	$+$	$+$	$+$	increasing on $(\sqrt{3}, \infty)$

(b) Local minimum values $f(\pm\sqrt{3}) = -9$, local maximum value $f(0) = 0$

(c) $f''(x) = 12x^2 - 12 = 12(x^2 - 1) > 0 \;\Leftrightarrow\; x^2 > 1 \;\Leftrightarrow\; |x| > 1 \;\Leftrightarrow\; x > 1$ or $x < -1$, so f is CU on $(-\infty, -1)$, $(1, \infty)$ and CD on $(-1, 1)$. Inflection points at $(\pm 1, -5)$

(d)

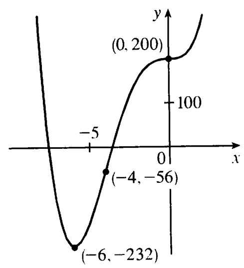


36. (a) $g(x) = 200 + 8x^3 + x^4 \;\Rightarrow\; g'(x) = 24x^2 + 4x^3 = 4x^2(6 + x) = 0$ when $x = -6$ and when $x = 0$. $g'(x) > 0 \;\Leftrightarrow\; x > -6\ (x \neq 0)$ and $g'(x) < 0 \;\Leftrightarrow\; x < -6$, so g is decreasing on $(-\infty, -6)$ and g is increasing on $(-6, \infty)$, with a horizontal tangent at $x = 0$.

(b) $g(-6) = -232$ is a local minimum value. There is no local maximum value.

(c) $g''(x) = 48x + 12x^2 = 12x(4 + x) = 0$ when $x = -4$ and when $x = 0$. $g''(x) > 0 \;\Leftrightarrow\; x < -4$ or $x > 0$ and $g''(x) < 0 \;\Leftrightarrow\; -4 < x < 0$, so g is CU on $(-\infty, -4)$ and $(0, \infty)$, and g is CD on $(-4, 0)$. Inflection points at $(-4, -56)$ and $(0, 200)$

(d)



37. (a) $h(x) = 3x^5 - 5x^3 + 3 \;\Rightarrow\; h'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 0$ when $x = 0, \pm 1$. Since $15x^2$ is nonnegative, $h'(x) > 0 \;\Leftrightarrow\; x^2 > 1 \;\Leftrightarrow\; |x| > 1 \;\Leftrightarrow\; x > 1$ or $x < -1$, so h is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$, with a horizontal tangent at $x = 0$.

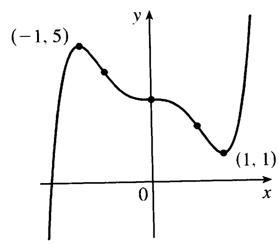
(b) Local maximum value $h(-1) = 5$, local minimum value $h(1) = 1$

(c) $h''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$

$$= 60x\left(x + \tfrac{1}{\sqrt{2}}\right)\left(x - \tfrac{1}{\sqrt{2}}\right) \;\Rightarrow$$

$h''(x) > 0$ when $x > \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}} < x < 0$, so h is CU on $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$ and CD on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(0, \frac{1}{\sqrt{2}}\right)$. Inflection points at $(0, 3)$ and $\left(\pm\frac{1}{\sqrt{2}}, 3 \mp \frac{7}{8}\sqrt{2}\right)$ [about $(-0.71, 4.24)$ and $(0.71, 1.76)$].

(d)



38. (a) $h(x) = (x^2 - 1)^3 \;\Rightarrow\; h'(x) = 6x(x^2-1)^2 \geq 0 \;\Leftrightarrow\; x > 0\ (x \neq 1)$, so h is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

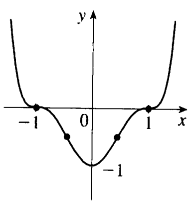
(b) $h(0) = -1$ is a local minimum value.

(c) $h''(x) = 6(x^2-1)^2 + 24x^2(x^2-1) = 6(x^2-1)(5x^2-1)$. The roots ± 1 and $\pm\frac{1}{\sqrt{5}}$ divide $\mathbb{R}$ into five intervals.

Interval	$x^2 - 1$	$5x^2 - 1$	$h''(x)$	Concavity
$x < -1$	+	+	+	upward
$-1 < x < -\frac{1}{\sqrt{5}}$	−	+	−	downward
$-\frac{1}{\sqrt{5}} < x < \frac{1}{\sqrt{5}}$	−	−	+	upward
$\frac{1}{\sqrt{5}} < x < 1$	−	+	−	downward
$x > 1$	+	+	+	upward

From the table, we see that h is CU on $(-\infty, -1)$, $\left(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ and $(1, \infty)$, and CD on $\left(-1, -\frac{1}{\sqrt{5}}\right)$ and $\left(\frac{1}{\sqrt{5}}, 1\right)$.
Inflection points at $(\pm 1, 0)$ and $\left(\pm\frac{1}{\sqrt{5}}, -\frac{64}{125}\right)$

(d)



39. (a) $A(x) = x\sqrt{x+3} \;\Rightarrow$

$$A'(x) = x \cdot \tfrac{1}{2}(x+3)^{-1/2} + \sqrt{x+3} \cdot 1 = \frac{x}{2\sqrt{x+3}} + \sqrt{x+3} = \frac{x + 2(x+3)}{2\sqrt{x+3}} = \frac{3x+6}{2\sqrt{x+3}}.$$

The domain of A is $[-3, \infty)$. $A'(x) > 0$ for $x > -2$ and $A'(x) < 0$ for $-3 < x < -2$, so A is increasing on $(-2, \infty)$ and decreasing on $(-3, -2)$.

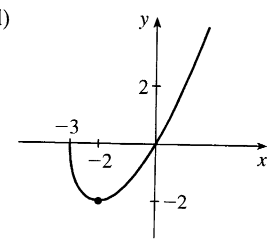
(b) $A(-2) = -2$ is a local minimum value.

(c)
$$A''(x) = \frac{2\sqrt{x+3} \cdot 3 - (3x+6) \cdot \dfrac{1}{\sqrt{x+3}}}{\left(2\sqrt{x+3}\right)^2}$$

$$= \frac{6(x+3) - (3x+6)}{4(x+3)^{3/2}} = \frac{3x+12}{4(x+3)^{3/2}} = \frac{3(x+4)}{4(x+3)^{3/2}}$$

$A''(x) > 0$ for all $x > -3$, so A is concave upward on $(-3, \infty)$. There is no inflection point.

(d)

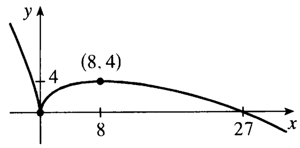


40. (a) $B(x) = 3x^{2/3} - x \;\Rightarrow\; B'(x) = 2x^{-1/3} - 1 = \dfrac{2}{\sqrt[3]{x}} - 1 = \dfrac{2 - \sqrt[3]{x}}{\sqrt[3]{x}}$. $B'(x) > 0$ if $0 < x < 8$ and $B'(x) < 0$ if $x < 0$ or $x > 8$, so B is decreasing on $(-\infty, 0)$ and $(8, \infty)$, and B is increasing on $(0, 8)$.

(b) $B(0) = 0$ is a local minimum value.
$B(8) = 4$ is a local maximum value.

(c) $B''(x) = -\frac{2}{3}x^{-4/3} = \dfrac{-2}{3x^{4/3}}$, so $B''(x) < 0$ for all $x \neq 0$. B is concave downward on $(-\infty, 0)$ and $(0, \infty)$. There is no inflection point.

(d)



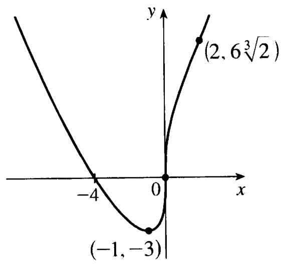
41. (a) $C(x) = x^{1/3}(x+4) = x^{4/3} + 4x^{1/3} \Rightarrow C'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x+1) = \dfrac{4(x+1)}{3\sqrt[3]{x^2}}$.
$C'(x) > 0$ if $-1 < x < 0$ or $x > 0$ and $C'(x) < 0$ for $x < -1$, so C is increasing on $(-1, \infty)$ and C is decreasing on $(-\infty, -1)$.

(b) $C(-1) = -3$ is a local minimum value.

(c) $C''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-5/3}(x-2) = \dfrac{4(x-2)}{9\sqrt[3]{x^5}}$.

$C''(x) < 0$ for $0 < x < 2$ and $C''(x) > 0$ for $x < 0$ and $x > 2$, so C is concave downward on $(0, 2)$ and concave upward on $(-\infty, 0)$ and $(2, \infty)$. There are inflection points at $(0, 0)$ and $(2, 6\sqrt[3]{2}) \approx (2, 7.56)$.

(d)



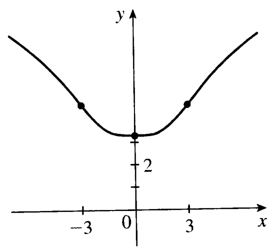
42. (a) $f(x) = \ln(x^4 + 27) \Rightarrow f'(x) = \dfrac{4x^3}{x^4 + 27}$. $f'(x) > 0$ if $x > 0$ and $f'(x) < 0$ if $x < 0$, so f is increasing on $(0, \infty)$ and f is decreasing on $(-\infty, 0)$.

(b) $f(0) = \ln 27 \approx 3.3$ is a local minimum value.

(c) $$f''(x) = \frac{(x^4+27)(12x^2) - 4x^3(4x^3)}{(x^4+27)^2} = \frac{4x^2\left[3(x^4+27) - 4x^4\right]}{(x^4+27)^2}$$
$$= \frac{4x^2(81 - x^4)}{(x^4+27)^2} = \frac{-4x^2(x^2+9)(x+3)(x-3)}{(x^4+27)^2}$$

$f''(x) > 0$ if $-3 < x < 0$ and $0 < x < 3$, and $f''(x) < 0$ if $x < -3$ or $x > 3$. Thus, f is concave upward on $(-3, 0)$ and $(0, 3)$ [hence on $(-3, 3)$] and f is concave downward on $(-\infty, -3)$ and $(3, \infty)$. There are inflection points at $(\pm 3, \ln 108) \approx (\pm 3, 4.68)$.

(d)



43. (a) $f(\theta) = 2\cos\theta - \cos 2\theta$, $0 \le \theta \le 2\pi$.
$f'(\theta) = -2\sin\theta + 2\sin 2\theta = -2\sin\theta + 2(2\sin\theta\cos\theta) = 2\sin\theta\,(2\cos\theta - 1)$.

Interval	$\sin\theta$	$2\cos\theta - 1$	$f'(\theta)$	f
$0 < \theta < \frac{\pi}{3}$	$+$	$+$	$+$	increasing on $(0, \frac{\pi}{3})$
$\frac{\pi}{3} < \theta < \pi$	$+$	$-$	$-$	decreasing on $(\frac{\pi}{3}, \pi)$
$\pi < \theta < \frac{5\pi}{3}$	$-$	$-$	$+$	increasing on $(\pi, \frac{5\pi}{3})$
$\frac{5\pi}{3} < \theta < 2\pi$	$-$	$+$	$-$	decreasing on $(\frac{5\pi}{3}, 2\pi)$

(b) $f(\frac{\pi}{3}) = \frac{3}{2}$ and $f(\frac{5\pi}{3}) = \frac{3}{2}$ are local maximum values and $f(\pi) = -3$ is a local minimum value.

(c) $f'(\theta) = -2\sin\theta + 2\sin 2\theta \;\Rightarrow$

$$f''(\theta) = -2\cos\theta + 4\cos 2\theta = -2\cos\theta + 4(2\cos^2\theta - 1) = 2(4\cos^2\theta - \cos\theta - 2)$$

$$f''(\theta) = 0 \;\Leftrightarrow\; \cos\theta = \frac{1 \pm \sqrt{33}}{8} \;\Leftrightarrow\; \theta = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$$

$$\Leftrightarrow\; \theta = \cos^{-1}\left(\frac{1+\sqrt{33}}{8}\right) \approx 0.5678,$$

$$2\pi - \cos^{-1}\left(\frac{1+\sqrt{33}}{8}\right) \approx 5.7154,\ \cos^{-1}\left(\frac{1-\sqrt{33}}{8}\right) \approx 2.2057,\ \text{or}\ 2\pi - \cos^{-1}\left(\frac{1-\sqrt{33}}{8}\right) \approx 4.0775.$$

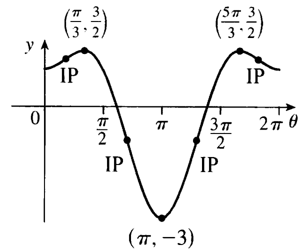
Denote these four values of θ by θ_1, θ_4, θ_2, and θ_3, respectively. Then f is CU on $(0, \theta_1)$, CD on (θ_1, θ_2), CU on (θ_2, θ_3), CD on (θ_3, θ_4), and CU on $(\theta_4, 2\pi)$. To find the *exact* y-coordinate for $\theta = \theta_1$, we have

$$f(\theta_1) = 2\cos\theta_1 - \cos 2\theta_1 = 2\cos\theta_1 - (2\cos^2\theta_1 - 1) = 2\left(\frac{1+\sqrt{33}}{8}\right) - 2\left(\frac{1+\sqrt{33}}{8}\right)^2 + 1$$

$$= \tfrac{1}{4} + \tfrac{1}{4}\sqrt{33} - \tfrac{1}{32} - \tfrac{1}{16}\sqrt{33} - \tfrac{33}{32} + 1 = \tfrac{3}{16} + \tfrac{3}{16}\sqrt{33} = \tfrac{3}{16}(1+\sqrt{33}) = y_1 \approx 1.26.$$

Similarly, $f(\theta_2) = \frac{3}{16}(1 - \sqrt{33}) = y_2 \approx -0.89$. So f has inflection points at (θ_1, y_1), (θ_2, y_2), (θ_3, y_2), and (θ_4, y_1).

(d)

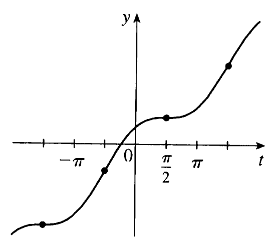


44. (a) $f(t) = t + \cos t$, $-2\pi \le t \le 2\pi \;\Rightarrow$

$f'(t) = 1 - \sin t \ge 0$ for all t and $f'(t) = 0$ when $\sin t = 1 \;\Leftrightarrow\; t = -\frac{3\pi}{2}$ or $\frac{\pi}{2}$, so f is increasing on $(-2\pi, 2\pi)$.

(b) No maximum or minimum

(d)



(c) $f''(t) = -\cos t > 0 \;\Leftrightarrow\; t \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, so f is CU on these intervals and CD on $\left(-2\pi, -\frac{3\pi}{2}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$. Points of inflection at $\left(\pm\frac{3\pi}{2}, \pm\frac{3\pi}{2}\right)$ and $\left(\pm\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$

45. $f(x) = \dfrac{x^2}{x^2 - 1} = \dfrac{x^2}{(x+1)(x-1)}$ has domain $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

(a) $\displaystyle\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2/x^2}{(x^2-1)/x^2} = \lim_{x\to\pm\infty} \frac{1}{1 - 1/x^2} = \frac{1}{1-0} = 1$, so $y = 1$ is a HA.

$\displaystyle\lim_{x\to -1^-} \frac{x^2}{x^2-1} = \infty$ since $x^2 \to 1$ and $(x^2 - 1) \to 0^+$ as $x \to -1^-$, so $x = -1$ is a VA.

$\displaystyle\lim_{x\to 1^+} \frac{x^2}{x^2-1} = \infty$ since $x^2 \to 1$ and $(x^2 - 1) \to 0^+$ as $x \to 1^+$, so $x = 1$ is a VA.

(b) $f(x) = \dfrac{x^2}{x^2-1} \;\Rightarrow\; f'(x) = \dfrac{(x^2-1)(2x) - x^2(2x)}{(x^2-1)^2} = \dfrac{2x[(x^2-1) - x^2]}{(x^2-1)^2} = \dfrac{-2x}{(x^2-1)^2}$. Since $(x^2-1)^2$ is positive for all x in the domain of f, the sign of the derivative is determined by the sign of $-2x$. Thus, $f'(x) > 0$ if $x < 0$ $(x \ne -1)$ and $f'(x) < 0$ if $x > 0$ $(x \ne 1)$. So f is increasing on $(-\infty, -1)$ and $(-1, 0)$, and f is decreasing on $(0, 1)$ and $(1, \infty)$.

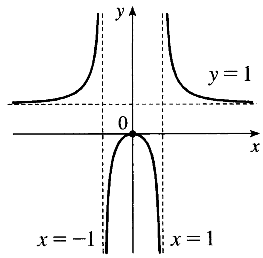
(c) $f'(x) = 0 \Rightarrow x = 0$ and $f(0) = 0$ is a local maximum value.

(d) $f''(x) = \dfrac{(x^2-1)^2(-2) - (-2x)\cdot 2(x^2-1)(2x)}{[(x^2-1)^2]^2}$

$= \dfrac{2(x^2-1)[-(x^2-1)+4x^2]}{(x^2-1)^4} = \dfrac{2(3x^2+1)}{(x^2-1)^3}$.

The sign of $f''(x)$ is determined by the denominator; that is, $f''(x) > 0$ if $|x| > 1$ and $f''(x) < 0$ if $|x| < 1$. Thus, f is CU on $(-\infty, -1)$ and $(1, \infty)$, and f is CD on $(-1, 1)$. There are no inflection points.

(e)



46. $f(x) = \dfrac{x^2}{(x-2)^2}$ has domain $(-\infty, 2) \cup (2, \infty)$.

(a) $\lim\limits_{x\to\pm\infty} \dfrac{x^2}{x^2-4x+4} = \lim\limits_{x\to\pm\infty} \dfrac{x^2/x^2}{(x^2-4x+4)/x^2} = \lim\limits_{x\to\pm\infty} \dfrac{1}{1-4/x+4/x^2} = \dfrac{1}{1-0+0} = 1$,

so $y = 1$ is a HA. $\lim\limits_{x\to 2^+} \dfrac{x^2}{(x-2)^2} = \infty$ since $x^2 \to 4$ and $(x-2)^2 \to 0^+$ as $x \to 2^+$, so $x = 2$ is a VA.

(b) $f(x) = \dfrac{x^2}{(x-2)^2} \Rightarrow f'(x) = \dfrac{(x-2)^2(2x) - x^2\cdot 2(x-2)}{[(x-2)^2]^2} = \dfrac{2x(x-2)[(x-2)-x]}{(x-2)^4} = \dfrac{-4x}{(x-2)^3}$.

$f'(x) > 0$ if $0 < x < 2$ and $f'(x) < 0$ if $x < 0$ or $x > 2$, so f is increasing on $(0, 2)$ and f is decreasing on $(-\infty, 0)$ and $(2, \infty)$.

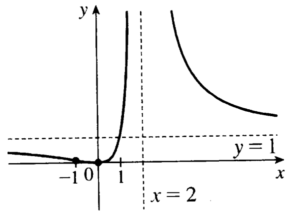
(c) $f(0) = 0$ is a local minimum value.

(d) $f''(x) = \dfrac{(x-2)^3(-4) - (-4x)\cdot 3(x-2)^2}{[(x-2)^3]^2}$

$= \dfrac{4(x-2)^2[-(x-2)+3x]}{(x-2)^6} = \dfrac{8(x+1)}{(x-2)^4}$

$f''(x) > 0$ if $x > -1$ $(x \neq 2)$ and $f''(x) < 0$ if $x < -1$. Thus, f is CU on $(-1, 2)$ and $(2, \infty)$, and f is CD on $(-\infty, -1)$. There is an inflection point at $\left(-1, \frac{1}{9}\right)$.

(e)



47. (a) $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+1} - x\right) = \infty$ and

$\lim\limits_{x\to\infty} \left(\sqrt{x^2+1} - x\right) = \lim\limits_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x^2+1}+x} = 0$, so $y = 0$ is a HA.

(b) $f(x) = \sqrt{x^2+1} - x \Rightarrow f'(x) = \dfrac{x}{\sqrt{x^2+1}} - 1$. Since $\dfrac{x}{\sqrt{x^2+1}} < 1$ for all x, $f'(x) < 0$, so f is decreasing on $\mathbb{R}$.

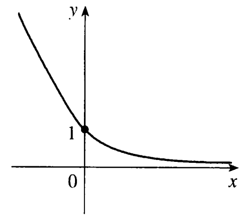
(c) No minimum or maximum

(d) $$f''(x) = \frac{(x^2+1)^{1/2}(1) - x \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x)}{\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{(x^2+1)^{1/2} - \dfrac{x^2}{(x^2+1)^{1/2}}}{x^2+1} = \frac{(x^2+1) - x^2}{(x^2+1)^{3/2}}$$

$$= \frac{1}{(x^2+1)^{3/2}} > 0, \text{ so } f \text{ is CU on } \mathbb{R}. \text{ No IP}$$

(e)



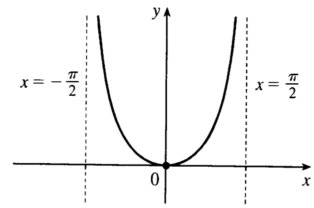
48. (a) $\lim\limits_{x\to\pi/2^-} x\tan x = \infty$ and $\lim\limits_{x\to-\pi/2^+} x\tan x = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA.

(b) $f(x) = x\tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. $f'(x) = x\sec^2 x + \tan x > 0$ $\Leftrightarrow$ $0 < x < \frac{\pi}{2}$, so f increases on $\left(0, \frac{\pi}{2}\right)$ and decreases on $\left(-\frac{\pi}{2}, 0\right)$.

(c) $f(0) = 0$ is a local minimum value.

(d) $f''(x) = 2\sec^2 x + 2x\,\tan x\,\sec^2 x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so f is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. No IP

(e)



49. $f(x) = \ln(1 - \ln x)$ is defined when $x > 0$ (so that $\ln x$ is defined) and $1 - \ln x > 0$ [so that $\ln(1 - \ln x)$ is defined]. The second condition is equivalent to $1 > \ln x$ $\Leftrightarrow$ $x < e$, so f has domain $(0, e)$.

(a) As $x \to 0^+$, $\ln x \to -\infty$, so $1 - \ln x \to \infty$ and $f(x) \to \infty$. As $x \to e^-$, $\ln x \to 1^-$, so $1 - \ln x \to 0^+$ and $f(x) \to -\infty$. Thus, $x = 0$ and $x = e$ are vertical asymptotes. There is no horizontal asymptote.

(b) $f'(x) = \dfrac{1}{1 - \ln x}\left(-\dfrac{1}{x}\right) = -\dfrac{1}{x(1 - \ln x)} < 0$ on $(0, e)$. Thus, f is decreasing on its domain, $(0, e)$.

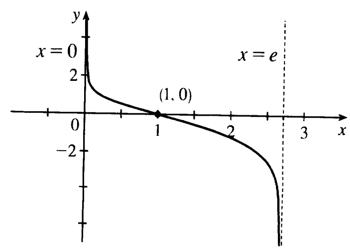
(c) $f'(x) \neq 0$ on $(0, e)$, so f has no local maximum or minimum value.

(d) $$f''(x) = -\frac{-\left[x(1-\ln x)\right]'}{\left[x(1-\ln x)\right]^2} = \frac{x(-1/x) + (1 - \ln x)}{x^2(1-\ln x)^2}$$

$$= -\frac{\ln x}{x^2(1-\ln x)^2}$$

so $f''(x) > 0$ $\Leftrightarrow$ $\ln x < 0$ $\Leftrightarrow$ $0 < x < 1$. Thus, f is CU on $(0, 1)$ and CD on $(1, e)$. There is an inflection point at $(1, 0)$.

(e)



50. $f(x) = \dfrac{e^x}{1 + e^x}$ has domain $\mathbb{R}$.

(a) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{e^x/e^x}{(1+e^x)/e^x} = \lim\limits_{x\to\infty} \dfrac{1}{e^{-x}+1} = \dfrac{1}{0+1} = 1$, so $y = 1$ is a HA.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{e^x}{1+e^x} = \dfrac{0}{1+0} = 0$, so $y = 0$ is a HA. No VA.

(b) $f'(x) = \dfrac{(1+e^x)e^x - e^x \cdot e^x}{(1+e^x)^2} = \dfrac{e^x}{(1+e^x)^2} > 0$ for all x. Thus, f is increasing on $\mathbb{R}$.

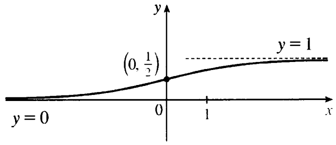
(c) There is no local maximum or minimum.

(d) $f''(x) = \dfrac{(1+e^x)^2\, e^x - e^x \cdot 2(1+e^x)\, e^x}{\left[(1+e^x)^2\right]^2}$

$= \dfrac{e^x(1+e^x)[(1+e^x) - 2e^x]}{(1+e^x)^4} = \dfrac{e^x(1-e^x)}{(1+e^x)^3}$

$f''(x) > 0 \iff 1 - e^x > 0 \iff x < 0$, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. There is an inflection point at $\left(0, \frac{1}{2}\right)$.

(e)



51. (a) $\lim\limits_{x\to\pm\infty} e^{-1/(x+1)} = 1$ since $-1/(x+1) \to 0$, so $y = 1$ is a HA. $\lim\limits_{x\to-1^+} e^{-1/(x+1)} = 0$ since $-1/(x+1) \to -\infty$, $\lim\limits_{x\to-1^-} e^{-1/(x+1)} = \infty$ since $-1/(x+1) \to \infty$, so $x = -1$ is a VA.

(b) $f(x) = e^{-1/(x+1)} \Rightarrow f'(x) = e^{-1/(x+1)}\left[-(-1)\dfrac{1}{(x+1)^2}\right]$ [Reciprocal Rule] $= e^{-1/(x+1)}/(x+1)^2$ $\Rightarrow f'(x) > 0$ for all x except -1, so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$.

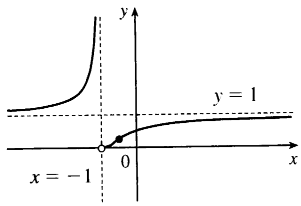
(c) No local maximum or minimum

(d) $f''(x) = \dfrac{(x+1)^2 e^{-1/(x+1)}\left[1/(x+1)^2\right] - e^{-1/(x+1)}\,[2(x+1)]}{[(x+1)^2]^2}$

$= \dfrac{e^{-1/(x+1)}\,[1 - (2x+2)]}{(x+1)^4} = -\dfrac{e^{-1/(x+1)}(2x+1)}{(x+1)^4} \Rightarrow$

$f''(x) > 0 \iff 2x + 1 < 0 \iff x < -\frac{1}{2}$, so f is CU on $(-\infty, -1)$ and $\left(-1, -\frac{1}{2}\right)$, and CD on $\left(-\frac{1}{2}, \infty\right)$. f has an IP at $\left(-\frac{1}{2}, e^{-2}\right)$.

(e)



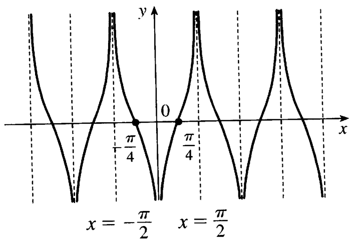
52. (a) f is periodic with period π, so we consider only $-\frac{\pi}{2} < x < \frac{\pi}{2}$. $\lim\limits_{x\to 0} \ln(\tan^2 x) = -\infty$, $\lim\limits_{x\to(\pi/2)^-} \ln(\tan^2 x) = \infty$, and $\lim\limits_{x\to(-\pi/2)^+} \ln(\tan^2 x) = \infty$, so $x = 0$, $x = \pm\frac{\pi}{2}$ are VA.

(b) $f(x) = \ln(\tan^2 x) \Rightarrow f'(x) = \dfrac{2\tan x \sec^2 x}{\tan^2 x} = 2\,\dfrac{\sec^2 x}{\tan x} > 0 \iff \tan x > 0 \iff 0 < x < \frac{\pi}{2}$, so f is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(-\frac{\pi}{2}, 0\right)$.

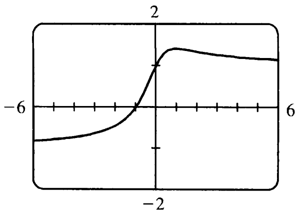
(c) No maximum or minimum

(d) $f'(x) = \dfrac{2}{\sin x \cos x} = \dfrac{4}{\sin 2x} \Rightarrow f''(x) = \dfrac{-8\cos 2x}{\sin^2 2x} < 0$ $\iff \cos 2x > 0 \iff -\frac{\pi}{4} < x < \frac{\pi}{4}$, so f is CD on $\left(-\frac{\pi}{4}, 0\right)$ and $\left(0, \frac{\pi}{4}\right)$, and CU on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. IP are $\left(\pm\frac{\pi}{4}, 0\right)$.

(e)



53. (a)



From the graph, we get an estimate of $f(1) \approx 1.41$ as a local maximum value, and no local minimum value.

$$f(x) = \frac{x+1}{\sqrt{x^2+1}} \quad \Rightarrow \quad f'(x) = \frac{1-x}{(x^2+1)^{3/2}}.$$

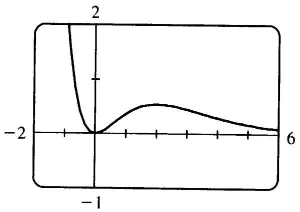
$f'(x) = 0 \quad \Leftrightarrow \quad x = 1.\ f(1) = \frac{2}{\sqrt{2}} = \sqrt{2}$ is the exact value.

(b) From the graph in part (a), f increases most rapidly somewhere between $x = -\frac{1}{2}$ and $x = -\frac{1}{4}$. To find the exact value, we need to find the maximum value of f', which we can do by finding the critical numbers of f'.

$$f''(x) = \frac{2x^2 - 3x - 1}{(x^2+1)^{5/2}} = 0 \quad \Leftrightarrow \quad x = \frac{3 \pm \sqrt{17}}{4}.\ x = \frac{3+\sqrt{17}}{4}$$ corresponds to the *minimum* value of f'.

The maximum value of f' is at $\left(\frac{3-\sqrt{17}}{4}, \sqrt{\frac{7}{6} - \frac{\sqrt{17}}{6}}\right) \approx (-0.28, 0.69)$.

54. (a)



Tracing the graph gives us estimates of $f(0) = 0$ for a local minimum value and $f(2) = 0.54$ for a local maximum value.

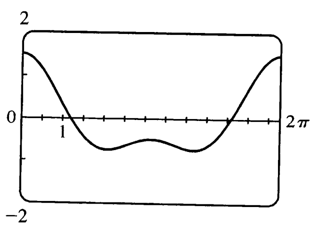
$f(x) = x^2 e^{-x} \quad \Rightarrow \quad f'(x) = xe^{-x}(2-x).\ f'(x) = 0 \quad \Leftrightarrow$

$x = 0$ or 2. $f(0) = 0$ and $f(2) = 4e^{-2}$ are the exact values.

(b) From the graph in part (a), f increases most rapidly around $x = \frac{3}{4}$. To find the exact value, we need to find the maximum value of f', which we can do by finding the critical numbers of f'. $f''(x) = e^{-x}(x^2 - 4x + 2) = 0$ $\Rightarrow \quad x = 2 \pm \sqrt{2}$. $x = 2 + \sqrt{2}$ corresponds to the *minimum* value of f'. The maximum value of f' is at $\left(2 - \sqrt{2}, \left(2 - \sqrt{2}\right)^2 e^{-2+\sqrt{2}}\right) \approx (0.59, 0.19)$.

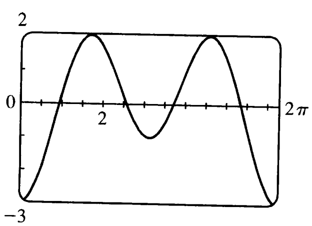
55. $f(x) = \cos x + \frac{1}{2}\cos 2x \quad \Rightarrow \quad f'(x) = -\sin x - \sin 2x \quad \Rightarrow \quad f''(x) = -\cos x - 2\cos 2x$

(a)



From the graph of f, it seems that f is CD on $(0, 1)$, CU on $(1, 2.5)$, CD on $(2.5, 3.7)$, CU on $(3.7, 5.3)$, and CD on $(5.3, 2\pi)$. The points of inflection appear to be at $(1, 0.4)$, $(2.5, -0.6)$, $(3.7, -0.6)$, and $(5.3, 0.4)$.

(b)



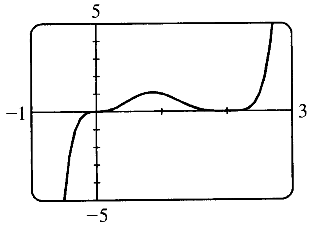
From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(0, 0.94)$, CU on $(0.94, 2.57)$, CD on $(2.57, 3.71)$, CU on $(3.71, 5.35)$, and CD on $(5.35, 2\pi)$. Refined estimates of the inflection points are $(0.94, 0.44)$, $(2.57, -0.63)$, $(3.71, -0.63)$, and $(5.35, 0.44)$.

56. $f(x) = x^3(x-2)^4 \quad \Rightarrow$

$$f'(x) = x^3 \cdot 4(x-2)^3 + (x-2)^4 \cdot 3x^2 = x^2(x-2)^3[4x + 3(x-2)] = x^2(x-2)^3(7x-6) \quad \Rightarrow$$

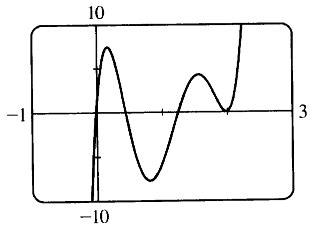
$$\begin{aligned} f''(x) &= (2x)(x-2)^3(7x-6) + x^2 \cdot 3(x-2)^2(7x-6) + x^2(x-2)^3(7) \\ &= x(x-2)^2[2(x-2)(7x-6) + 3x(7x-6) + 7x(x-2)] \\ &= x(x-2)^2[42x^2 - 72x + 24] = 6x(x-2)^2(7x^2 - 12x + 4) \end{aligned}$$

(a)



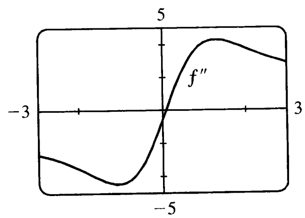
From the graph of f, it seems that f is CD on $(-\infty, 0)$, CU on $(0, 0.5)$, CD on $(0.5, 1.3)$, and CU on $(1.3, \infty)$. The points of inflection appear to be at $(0, 0)$, $(0.5, 0.5)$, and $(1.3, 0.6)$.

(b)

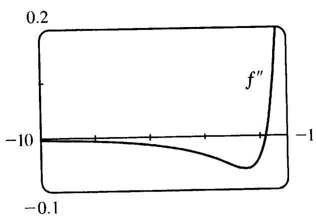


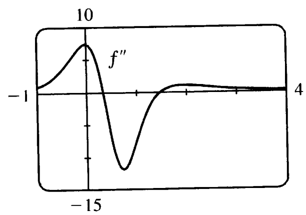
From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(-\infty, 0)$, CU on $(0, 0.45)$, CD on $(0.45, 1.26)$, and CU on $(1.26, \infty)$. Refined estimates of the inflection points are $(0, 0)$, $(0.45, 0.53)$, and $(1.26, 0.60)$.

57. In Maple, we define f and then use the command `plot(diff(diff(f,x),x),x=-3..3);`. In Mathematica, we define f and then use `Plot[Dt[Dt[f,x],x],{x,-3,3}]`.We see that $f'' > 0$ for $x > 0.1$ and $f'' < 0$ for $x < 0.1$. So f is concave up on $(0.1, \infty)$ and concave down on $(-\infty, 0.1)$.



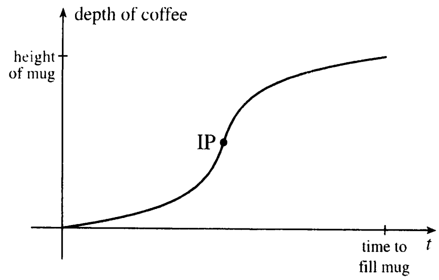
58. It appears that f'' is positive (and thus f is concave up) on $(-1.8, 0.3)$ and $(1.5, \infty)$ and negative (so f is concave down) on $(-\infty, -1.8)$ and $(0.3, 1.5)$.



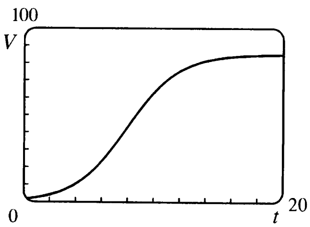


59. Most students learn more in the third hour of studying than in the eighth hour, so $K(3) - K(2)$ is larger than $K(8) - K(7)$. In other words, as you begin studying for a test, the rate of knowledge gain is large and then starts to taper off, so $K'(t)$ decreases and the graph of K is concave downward.

60. At first the depth increases slowly because the base of the mug is wide. But as the mug narrows, the coffee rises more quickly. Thus, the depth d increases at an increasing rate and its graph is concave upward. The rate of increase of d has a maximum where the mug is narrowest; that is, when the mug is half full. It is there that the inflection point (IP) occurs. Then the rate of increase of d starts to decrease as the mug widens and the graph becomes concave down.



61.



From the graph, we estimate that the most rapid increase in the percentage of households in the United States with at least one VCR occurs at about $t = 8$. To maximize the first derivative, we need to determine the values for which the second derivative is 0. We'll use $V(t) = \dfrac{a}{1 + be^{ct}}$, and substitute $a = 85$, $b = 53$, and $c = -0.5$ later.

$$V'(t) = -\frac{a(bce^{ct})}{(1 + be^{ct})^2} \quad \text{[by the Reciprocal Rule]} \quad \text{and}$$

$$V''(t) = -abc \cdot \frac{(1 + be^{ct})^2 \cdot ce^{ct} - e^{ct} \cdot 2(1 + be^{ct}) \cdot bce^{ct}}{\left[(1 + be^{ct})^2\right]^2}$$

$$= \frac{-abc \cdot ce^{ct}(1 + be^{ct})[(1 + be^{ct}) - 2be^{ct}]}{(1 + be^{ct})^4} = \frac{-abc^2e^{ct}(1 - be^{ct})}{(1 + be^{ct})^3}$$

So $V''(t) = 0 \;\Leftrightarrow\; 1 = be^{ct} \;\Leftrightarrow\; e^{ct} = 1/b$. Now graph $y = e^{-0.5t}$ and $y = \frac{1}{53}$. These graphs intersect at $t \approx 7.94$ years, which corresponds to roughly midyear 1988. [Alternatively, we could use the rootfinder on a calculator to solve $e^{-0.5t} = \frac{1}{53}$. Or, if you have already studied logarithms, you can solve $e^{ct} = 1/b$ as follows: $ct = \ln(1/b) \;\Leftrightarrow\; t = (1/c)\ln(1/b) = -2\ln\frac{1}{53} \approx 7.94$ years.

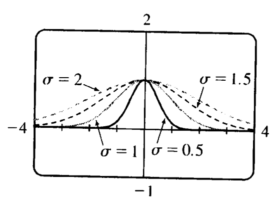
62. (a) As $|x| \to \infty$, $t = -x^2/(2\sigma^2) \to -\infty$, and $e^t \to 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t. Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \;\Rightarrow$ $f'(x) = e^{-x^2/(2\sigma^2)}(-x/\sigma^2)$. $f'(x) = 0 \;\Leftrightarrow\; x = 0$. Because f' changes from positive to negative at $x = 0$, $f(0) = 1$ is a local maximum. For inflection points, we find

$$f''(x) = -\frac{1}{\sigma^2}\left[e^{-x^2/(2\sigma^2)} \cdot 1 + xe^{-x^2/(2\sigma^2)}(-x/\sigma^2)\right] = \frac{-1}{\sigma^2}e^{-x^2/(2\sigma^2)}(1 - x^2/\sigma^2).$$

$f''(x) = 0 \;\Leftrightarrow\; x^2 = \sigma^2 \;\Leftrightarrow\; x = \pm\sigma$. $f''(x) < 0 \;\Leftrightarrow\; x^2 < \sigma^2 \;\Leftrightarrow\; -\sigma < x < \sigma$. So f is CD on $(-\sigma, \sigma)$ and CU on $(-\infty, -\sigma)$ and (σ, ∞). IP at $\left(\pm\sigma, e^{-1/2}\right)$.

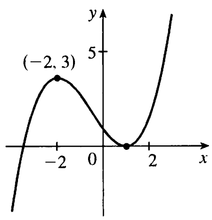
(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y-axis as σ increases.

(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x-axis.

63. $f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c$. We are given that $f(1) = 0$ and $f(-2) = 3$, so $f(1) = a + b + c + d = 0$ and $f(-2) = -8a + 4b - 2c + d = 3$. Also $f'(1) = 3a + 2b + c = 0$ and $f'(-2) = 12a - 4b + c = 0$ by Fermat's Theorem. Solving these four equations, we get $a = \frac{2}{9}, b = \frac{1}{3}, c = -\frac{4}{3}, d = \frac{7}{9}$, so the function is $f(x) = \frac{1}{9}(2x^3 + 3x^2 - 12x + 7)$.



64. $f(x) = axe^{bx^2} \Rightarrow f'(x) = a\left[xe^{bx^2} \cdot 2bx + e^{bx^2} \cdot 1\right] = ae^{bx^2}(2bx^2 + 1)$. For $f(2) = 1$ to be a maximum value, we must have $f'(2) = 0$. $f(2) = 1 \Rightarrow 1 = 2ae^{4b}$ and $f'(2) = 0 \Rightarrow 0 = (8b + 1)ae^{4b}$. So $8b + 1 = 0$ $[a \neq 0] \Rightarrow b = -\frac{1}{8}$ and now $1 = 2ae^{-1/2} \Rightarrow a = \sqrt{e}/2$.

65. Suppose that f is differentiable on an interval I and $f'(x) > 0$ for all x in I except $x = c$. To show that f is increasing on I, let x_1, x_2 be two numbers in I with $x_1 < x_2$.

Case 1 $x_1 < x_2 < c$. Let J be the interval $\{x \in I \mid x < c\}$. By applying the Increasing/Decreasing Test to f on J, we see that f is increasing on J, so $f(x_1) < f(x_2)$.

Case 2 $c < x_1 < x_2$. Apply the Increasing/Decreasing Test to f on $K = \{x \in I \mid x > c\}$.

Case 3 $x_1 < x_2 = c$. Apply the proof of the Increasing/Decreasing Test, using the Mean Value Theorem (MVT) on the interval $[x_1, x_2]$ and noting that the MVT does not require f to be differentiable at the endpoints of $[x_1, x_2]$.

Case 4 $c = x_1 < x_2$. Same proof as in Case 3.

Case 5 $x_1 < c < x_2$. By Cases 3 and 4, f is increasing on $[x_1, c]$ and on $[c, x_2]$, so $f(x_1) < f(c) < f(x_2)$.

In all cases, we have shown that $f(x_1) < f(x_2)$. Since x_1, x_2 were any numbers in I with $x_1 < x_2$, we have shown that f is increasing on I.

66. (a) We will make use of the converse of the Concavity Test (along with the stated assumptions); that is, if f is concave upward on I, then $f'' > 0$ on I. If f and g are CU on I, then $f'' > 0$ and $g'' > 0$ on I, so $(f + g)'' = f'' + g'' > 0$ on $I \Rightarrow f + g$ is CU on I.

(b) Since f is positive and CU on I, $f > 0$ and $f'' > 0$ on I. So $g(x) = [f(x)]^2 \Rightarrow g' = 2ff' \Rightarrow$ $g'' = 2f'f' + 2ff'' = 2(f')^2 + 2ff'' > 0 \Rightarrow g$ is CU on I.

67. (a) Since f and g are positive, increasing, and CU on I with f'' and g'' never equal to 0, we have $f > 0$, $f' \geq 0$, $f'' > 0$, $g > 0$, $g' \geq 0$, $g'' > 0$ on I. Then $(fg)' = f'g + fg' \Rightarrow$ $(fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \Rightarrow fg$ is CU on I.

(b) In part (a), if f and g are both decreasing instead of increasing, then $f' \leq 0$ and $g' \leq 0$ on I, so we still have $2f'g' \geq 0$ on I. Thus, $(fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \Rightarrow fg$ is CU on I as in part (a).

(c) Suppose f is increasing and g is decreasing [with f and g positive and CU]. Then $f' \geq 0$ and $g' \leq 0$ on I, so $2f'g' \leq 0$ on I and the argument in parts (a) and (b) fails.

Example 1. $I = (0, \infty)$, $f(x) = x^3$, $g(x) = 1/x$. Then $(fg)(x) = x^2$, so $(fg)'(x) = 2x$ and $(fg)''(x) = 2 > 0$ on I. Thus, fg is CU on I.

Example 2. $I = (0, \infty)$, $f(x) = 4x\sqrt{x}$, $g(x) = 1/x$. Then $(fg)(x) = 4\sqrt{x}$, so $(fg)'(x) = 2/\sqrt{x}$ and $(fg)''(x) = -1/\sqrt{x^3} < 0$ on I. Thus, fg is CD on I.

Example 3. $I = (0, \infty)$, $f(x) = x^2$, $g(x) = 1/x$. Thus, $(fg)(x) = x$, so fg is linear on I.

68. Since f and g are CU on $(-\infty, \infty)$, $f'' > 0$ and $g'' > 0$ on $(-\infty, \infty)$.

$h(x) = f(g(x)) \;\Rightarrow\; h'(x) = f'(g(x))g'(x) \;\Rightarrow$

$h''(x) = f''(g(x))g'(x)g'(x) + f'(g(x))g''(x) = f''(g(x))[g'(x)]^2 + f'(g(x))g''(x) > 0$ if $f' > 0$.

So h is CU if f is increasing.

69. $f(x) = \tan x - x \;\Rightarrow\; f'(x) = \sec^2 x - 1 > 0$ for $0 < x < \frac{\pi}{2}$ since $\sec^2 x > 1$ for $0 < x < \frac{\pi}{2}$. So f is increasing on $\left(0, \frac{\pi}{2}\right)$. Thus, $f(x) > f(0) = 0$ for $0 < x < \frac{\pi}{2} \;\Rightarrow\; \tan x - x > 0 \;\Rightarrow\; \tan x > x$ for $0 < x < \frac{\pi}{2}$.

70. (a) Let $f(x) = e^x - 1 - x$. Now $f(0) = e^0 - 1 = 0$, and for $x \geq 0$, we have $f'(x) = e^x - 1 \geq 0$. Now, since $f(0) = 0$ and f is increasing on $[0, \infty)$, $f(x) \geq 0$ for $x \geq 0 \;\Rightarrow\; e^x - 1 - x \geq 0 \;\Rightarrow\; e^x \geq 1 + x$.

(b) Let $f(x) = e^x - 1 - x - \frac{1}{2}x^2$. Thus, $f'(x) = e^x - 1 - x$, which is positive for $x \geq 0$ by part (a). Thus, $f(x)$ is increasing on $(0, \infty)$, so on that interval, $0 = f(0) \leq f(x) = e^x - 1 - x - \frac{1}{2}x^2 \;\Rightarrow\; e^x \geq 1 + x + \frac{1}{2}x^2$.

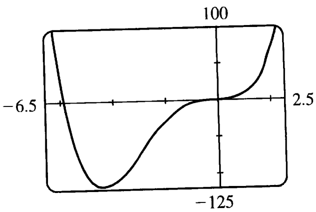
(c) By part (a), the result holds for $n = 1$. Suppose that $e^x \geq 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^k}{k!}$ for $x \geq 0$.

Let $f(x) = e^x - 1 - x - \dfrac{x^2}{2!} - \cdots - \dfrac{x^k}{k!} - \dfrac{x^{k+1}}{(k+1)!}$. Then $f'(x) = e^x - 1 - x - \cdots - \dfrac{x^k}{k!} \geq 0$ by assumption. Hence, $f(x)$ is increasing on $(0, \infty)$. So $0 \leq x$ implies that

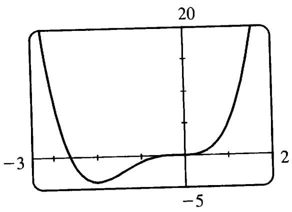
$0 = f(0) \leq f(x) = e^x - 1 - x - \cdots - \dfrac{x^k}{k!} - \dfrac{x^{k+1}}{(k+1)!}$, and hence $e^x \geq 1 + x + \cdots + \dfrac{x^k}{k!} + \dfrac{x^{k+1}}{(k+1)!}$ for $x \geq 0$. Therefore, for $x \geq 0$, $e^x \geq 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!}$ for every positive integer n, by mathematical induction.

71. Let the cubic function be $f(x) = ax^3 + bx^2 + cx + d \;\Rightarrow\; f'(x) = 3ax^2 + 2bx + c \;\Rightarrow\; f''(x) = 6ax + 2b$. So f is CU when $6ax + 2b > 0 \;\Leftrightarrow\; x > -b/(3a)$, CD when $x < -b/(3a)$, and so the only point of inflection occurs when $x = -b/(3a)$. If the graph has three x-intercepts x_1, x_2 and x_3, then the expression for $f(x)$ must factor as $f(x) = a(x - x_1)(x - x_2)(x - x_3)$. Multiplying these factors together gives us $f(x) = a\left[x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3\right]$. Equating the coefficients of the x^2-terms for the two forms of f gives us $b = -a(x_1 + x_2 + x_3)$. Hence, the x-coordinate of the point of inflection is $-\dfrac{b}{3a} = -\dfrac{-a(x_1 + x_2 + x_3)}{3a} = \dfrac{x_1 + x_2 + x_3}{3}$.

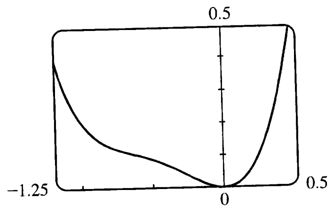
72. $P(x) = x^4 + cx^3 + x^2 \quad \Rightarrow \quad P'(x) = 4x^3 + 3cx^2 + 2x \quad \Rightarrow \quad P''(x) = 12x^2 + 6cx + 2$. The graph of $P''(x)$ is a parabola. If $P''(x)$ has two roots, then it changes sign twice and so has two inflection points. This happens when the discriminant of $P''(x)$ is positive, that is, $(6c)^2 - 4 \cdot 12 \cdot 2 > 0 \quad \Leftrightarrow \quad 36c^2 - 96 > 0 \quad \Leftrightarrow$ $|c| > \frac{2\sqrt{6}}{3} \approx 1.63$. If $36c^2 - 96 = 0 \quad \Leftrightarrow \quad c = \pm\frac{2\sqrt{6}}{3}$, $P''(x)$ is 0 at one point, but there is still no inflection point since $P''(x)$ never changes sign, and if $36c^2 - 96 < 0 \quad \Leftrightarrow \quad |c| < \frac{2\sqrt{6}}{3}$, then $P''(x)$ never changes sign, and so there is no inflection point.



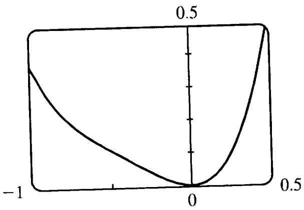
$c = 6$



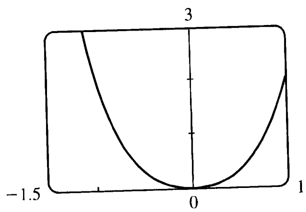
$c = 3$



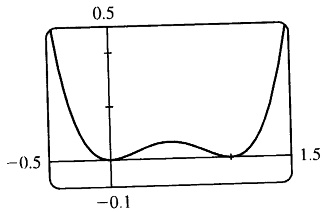
$c = 1.8$



$c = \frac{2\sqrt{6}}{3}$



$c = 0$



$c = -2$

For large positive c, the graph of f has two inflection points and a large dip to the left of the y-axis. As c decreases, the graph of f becomes flatter for $x < 0$, and eventually the dip rises above the x-axis, and then disappears entirely, along with the inflection points. As c continues to decrease, the dip and the inflection points reappear, to the right of the origin.

73. By hypothesis $g = f'$ is differentiable on an open interval containing c. Since $(c, f(c))$ is a point of inflection, the concavity changes at $x = c$, so $f''(x)$ changes signs at $x = c$. Hence, by the First Derivative Test, f' has a local extremum at $x = c$. Thus, by Fermat's Theorem $f''(c) = 0$.

74. $f(x) = x^4 \quad \Rightarrow \quad f'(x) = 4x^3 \quad \Rightarrow \quad f''(x) = 12x^2 \quad \Rightarrow \quad f''(0) = 0$. For $x < 0$, $f''(x) > 0$, so f is CU on $(-\infty, 0)$; for $x > 0$, $f''(x) > 0$, so f is also CU on $(0, \infty)$. Since f does not change concavity at 0, $(0, 0)$ is not an inflection point.

75. Using the fact that $|x| = \sqrt{x^2}$, we have that $g(x) = x\sqrt{x^2} \quad \Rightarrow \quad g'(x) = \sqrt{x^2} + \sqrt{x^2} = 2\sqrt{x^2} = 2|x| \quad \Rightarrow$ $g''(x) = 2x(x^2)^{-1/2} = \dfrac{2x}{|x|} < 0$ for $x < 0$ and $g''(x) > 0$ for $x > 0$, so $(0, 0)$ is an inflection point. But $g''(0)$ does not exist.

76. There must exist some interval containing c on which f''' is positive, since $f'''(c)$ is positive and f''' is continuous. On this interval, f'' is increasing (since f''' is positive), so $f'' = (f')'$ changes from negative to positive at c. So by the First Derivative Test, f' has a local minimum at $x = c$ and thus cannot change sign there, so f has no maximum or minimum at c. But since f'' changes from negative to positive at c, f has a point of inflection at c (it changes from concave down to concave up).

4.4 Indeterminate Forms and L'Hospital's Rule

The use of l'Hospital's Rule is indicated by an H above the equal sign: $\overset{\text{H}}{=}$.

1. (a) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is an indeterminate form of type $\dfrac{0}{0}$.

(b) $\lim\limits_{x\to a} \dfrac{f(x)}{p(x)} = 0$ because the numerator approaches 0 while the denominator becomes large.

(c) $\lim\limits_{x\to a} \dfrac{h(x)}{p(x)} = 0$ because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim\limits_{x\to a} p(x) = \infty$ and $f(x) \to 0$ through positive values, then $\lim\limits_{x\to a} \dfrac{p(x)}{f(x)} = \infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x^2$.] If $f(x) \to 0$ through negative values, then $\lim\limits_{x\to a} \dfrac{p(x)}{f(x)} = -\infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = -x^2$.] If $f(x) \to 0$ through both positive and negative values, then the limit might not exist. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x$.]

(e) $\lim\limits_{x\to a} \dfrac{p(x)}{q(x)}$ is an indeterminate form of type $\dfrac{\infty}{\infty}$.

2. (a) $\lim\limits_{x\to a} [f(x)p(x)]$ is an indeterminate form of type $0 \cdot \infty$.

(b) When x is near a, $p(x)$ is large and $h(x)$ is near 1, so $h(x)p(x)$ is large. Thus, $\lim\limits_{x\to a} [h(x)p(x)] = \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x)q(x)$ is large. Thus, $\lim\limits_{x\to a} [p(x)q(x)] = \infty$.

3. (a) When x is near a, $f(x)$ is near 0 and $p(x)$ is large, so $f(x) - p(x)$ is large negative. Thus, $\lim\limits_{x\to a} [f(x) - p(x)] = -\infty$.

(b) $\lim\limits_{x\to a} [p(x) - q(x)]$ is an indeterminate form of type $\infty - \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x) + q(x)$ is large. Thus, $\lim\limits_{x\to a} [p(x) + q(x)] = \infty$.

4. (a) $\lim_{x\to a} [f(x)]^{g(x)}$ is an indeterminate form of type 0^0.

(b) If $y = [f(x)]^{p(x)}$, then $\ln y = p(x) \ln f(x)$. When x is near a, $p(x) \to \infty$ and $\ln f(x) \to -\infty$, so $\ln y \to -\infty$. Therefore, $\lim_{x\to a} [f(x)]^{p(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = 0$, provided f^p is defined.

(c) $\lim_{x\to a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞.

(d) $\lim_{x\to a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0.

(e) If $y = [p(x)]^{q(x)}$, then $\ln y = q(x) \ln p(x)$. When x is near a, $q(x) \to \infty$ and $\ln p(x) \to \infty$, so $\ln y \to \infty$. Therefore, $\lim_{x\to a} [p(x)]^{q(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = \infty$.

(f) $\lim_{x\to a} \sqrt[q(x)]{p(x)} = \lim_{x\to a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0.

5. This limit has the form $\frac{0}{0}$. We can simply factor the numerator to evaluate this limit.

$$\lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} \frac{(x+1)(x-1)}{x+1} = \lim_{x\to -1} (x-1) = -2$$

6. $\lim_{x\to -2} \dfrac{x+2}{x^2+3x+2} = \lim_{x\to -2} \dfrac{x+2}{(x+1)(x+2)} = \lim_{x\to -2} \dfrac{1}{x+1} = -1$

7. This limit has the form $\frac{0}{0}$. $\lim_{x\to 1} \dfrac{x^9-1}{x^5-1} \overset{\text{H}}{=} \lim_{x\to 1} \dfrac{9x^8}{5x^4} = \dfrac{9}{5} \lim_{x\to 1} x^4 = \dfrac{9}{5}(1) = \dfrac{9}{5}$

8. $\lim_{x\to 1} \dfrac{x^a-1}{x^b-1} \overset{\text{H}}{=} \lim_{x\to 1} \dfrac{ax^{a-1}}{bx^{b-1}} = \dfrac{a}{b}$

9. This limit has the form $\frac{0}{0}$. $\lim_{x\to (\pi/2)^+} \dfrac{\cos x}{1-\sin x} \overset{\text{H}}{=} \lim_{x\to (\pi/2)^+} \dfrac{-\sin x}{-\cos x} = \lim_{x\to (\pi/2)^+} \tan x = -\infty$.

10. $\lim_{x\to 0} \dfrac{x+\tan x}{\sin x} \overset{\text{H}}{=} \lim_{x\to 0} \dfrac{1+\sec^2 x}{\cos x} = \dfrac{1+1^2}{1} = 2$

11. This limit has the form $\frac{0}{0}$. $\lim_{t\to 0} \dfrac{e^t-1}{t^3} \overset{\text{H}}{=} \lim_{t\to 0} \dfrac{e^t}{3t^2} = \infty$ since $e^t \to 1$ and $3t^2 \to 0^+$ as $t \to 0$.

12. $\lim_{t\to 0} \dfrac{e^{3t}-1}{t} \overset{\text{H}}{=} \lim_{t\to 0} \dfrac{3e^{3t}}{1} = 3$

13. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0} \dfrac{\tan px}{\tan qx} \overset{\text{H}}{=} \lim_{x\to 0} \dfrac{p\sec^2 px}{q\sec^2 qx} = \dfrac{p(1)^2}{q(1)^2} = \dfrac{p}{q}$

14. $\lim_{\theta\to \pi/2} \dfrac{1-\sin\theta}{\csc\theta} = \dfrac{0}{1} = 0$. L'Hospital's Rule does not apply.

15. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to \infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to \infty} \dfrac{1/x}{1} = 0$

16. $\lim_{x\to \infty} \dfrac{e^x}{x} \overset{\text{H}}{=} \lim_{x\to \infty} \dfrac{e^x}{1} = \lim_{x\to \infty} e^x = \infty$

17. $\lim_{x\to 0^+} [(\ln x)/x] = -\infty$ since $\ln x \to -\infty$ as $x \to 0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.

18. $\lim_{x\to \infty} \dfrac{\ln \ln x}{x} \overset{\text{H}}{=} \lim_{x\to \infty} \dfrac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1} = \lim_{x\to \infty} \dfrac{1}{x\ln x} = 0$

19. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to 0}\dfrac{5^t-3^t}{t}\overset{\text{H}}{=}\lim\limits_{t\to 0}\dfrac{5^t\ln 5-3^t\ln 3}{1}=\ln 5-\ln 3=\ln\frac{5}{3}$

20. $\lim\limits_{x\to 1}\dfrac{\ln x}{\sin\pi x}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{1/x}{\pi\cos\pi x}=\dfrac{1}{\pi(-1)}=-\dfrac{1}{\pi}$

21. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x-1}{2x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x}{2}=\dfrac{1}{2}$

22. $\lim\limits_{x\to 0}\dfrac{e^x-1-x-x^2/2}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x-1-x}{3x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x-1}{6x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x}{6}=\dfrac{1}{6}$

23. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{e^x}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{3x^2}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{6x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{6}=\infty$

24. $\lim\limits_{x\to 0}\dfrac{\sin x}{\sinh x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\cos x}{\cosh x}=\dfrac{1}{1}=1$

25. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x}{x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{1/\sqrt{1-x^2}}{1}=\lim\limits_{x\to 0}\dfrac{1}{\sqrt{1-x^2}}=\dfrac{1}{1}=1$

26. $\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\cos x-1}{3x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{-\sin x}{6x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{-\cos x}{6}=-\dfrac{1}{6}$

27. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\sin x}{2x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\cos x}{2}=\dfrac{1}{2}$

28. $\lim\limits_{x\to\infty}\dfrac{(\ln x)^2}{x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{2(\ln x)(1/x)}{1}=2\lim\limits_{x\to\infty}\dfrac{\ln x}{x}\overset{\text{H}}{=}2\lim\limits_{x\to\infty}\dfrac{1/x}{1}=2(0)=0$

29. $\lim\limits_{x\to 0}\dfrac{x+\sin x}{x+\cos x}=\dfrac{0+0}{0+1}=\dfrac{0}{1}=0$. L'Hospital's Rule does not apply.

30. $\lim\limits_{x\to 0}\dfrac{\cos mx-\cos nx}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{-m\sin mx+n\sin nx}{2x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{-m^2\cos mx+n^2\cos nx}{2}=\frac{1}{2}(n^2-m^2)$

31. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{x}{\ln(1+2e^x)}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1}{\dfrac{1}{1+2e^x}\cdot 2e^x}=\lim\limits_{x\to\infty}\dfrac{1+2e^x}{2e^x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{2e^x}{2e^x}=1$

32. $\lim\limits_{x\to 0}\dfrac{x}{\tan^{-1}(4x)}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{1}{\dfrac{1}{1+(4x)^2}\cdot 4}=\lim\limits_{x\to 0}\dfrac{1+16x^2}{4}=\dfrac{1}{4}$

33. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1}\dfrac{1-x+\ln x}{1+\cos\pi x}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{-1+1/x}{-\pi\sin\pi x}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{-1/x^2}{-\pi^2\cos\pi x}=\dfrac{-1}{-\pi^2(-1)}=-\dfrac{1}{\pi^2}$

34. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+2}}{\sqrt{2x^2+1}}=\lim\limits_{x\to\infty}\sqrt{\dfrac{x^2+2}{2x^2+1}}=\sqrt{\lim\limits_{x\to\infty}\dfrac{x^2+2}{2x^2+1}}=\sqrt{\lim\limits_{x\to\infty}\dfrac{1+2/x^2}{2+1/x^2}}=\sqrt{\dfrac{1}{2}}$

35. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1}\dfrac{x^a-ax+a-1}{(x-1)^2}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{ax^{a-1}-a}{2(x-1)}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{a(a-1)x^{a-2}}{2}=\dfrac{a(a-1)}{2}$

36. $\lim\limits_{x\to 0}\dfrac{1-e^{-2x}}{\sec x}=\dfrac{1-1}{1}=0$. L'Hospital's Rule does not apply.

37. This limit has the form $0\cdot(-\infty)$. We need to write this product as a quotient, but keep in mind that we will have to differentiate both the numerator and the denominator. If we differentiate $\dfrac{1}{\ln x}$, we get a complicated expression that results in a more difficult limit. Instead we write the quotient as $\dfrac{\ln x}{x^{-1/2}}$.

$$\lim_{x\to 0^+}\sqrt{x}\ln x=\lim_{x\to 0^+}\frac{\ln x}{x^{-1/2}}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{1/x}{-\frac{1}{2}x^{-3/2}}\cdot\frac{-2x^{3/2}}{-2x^{3/2}}=\lim_{x\to 0^+}(-2\sqrt{x})=0$$

38. $\lim\limits_{x\to-\infty} x^2e^x = \lim\limits_{x\to-\infty} \dfrac{x^2}{e^{-x}} \overset{\text{H}}{=} \lim\limits_{x\to-\infty} \dfrac{2x}{-e^{-x}} \overset{\text{H}}{=} \lim\limits_{x\to-\infty} \dfrac{2}{e^{-x}} = \lim\limits_{x\to-\infty} 2e^x = 0$

39. This limit has the form $\infty \cdot 0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x\to0} \cot 2x \sin 6x = \lim_{x\to0} \frac{\sin 6x}{\tan 2x} \overset{\text{H}}{=} \lim_{x\to0} \frac{6\cos 6x}{2\sec^2 2x} = \frac{6(1)}{2(1)^2} = 3$$

40.
$$\lim_{x\to0^+} \sin x \ln x = \lim_{x\to0^+} \frac{\ln x}{\csc x} \overset{\text{H}}{=} \lim_{x\to0^+} \frac{1/x}{-\csc x \cot x} = -\lim_{x\to0^+}\left(\frac{\sin x}{x}\cdot \tan x\right)$$
$$= -\left(\lim_{x\to0^+}\frac{\sin x}{x}\right)\left(\lim_{x\to0^+}\tan x\right) = -1\cdot 0 = 0$$

41. This limit has the form $\infty \cdot 0$. $\lim\limits_{x\to\infty} x^3 e^{-x^2} = \lim\limits_{x\to\infty} \dfrac{x^3}{e^{x^2}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{3x^2}{2xe^{x^2}} = \lim\limits_{x\to\infty} \dfrac{3x}{2e^{x^2}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{3}{4xe^{x^2}} = 0$

42. $\lim\limits_{x\to\pi/4} (1-\tan x)\sec x = (1-1)\sqrt{2} = 0$. L'Hospital's Rule does not apply.

43. This limit has the form $0 \cdot (-\infty)$.

$$\lim_{x\to1^+} \ln x \tan(\pi x/2) = \lim_{x\to1^+} \frac{\ln x}{\cot(\pi x/2)} \overset{\text{H}}{=} \lim_{x\to1^+} \frac{1/x}{(-\pi/2)\csc^2(\pi x/2)} = \frac{1}{(-\pi/2)(1)^2} = -\frac{2}{\pi}$$

44. $\lim\limits_{x\to\infty} x\tan(1/x) = \lim\limits_{x\to\infty} \dfrac{\tan(1/x)}{1/x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{\sec^2(1/x)(-1/x^2)}{-1/x^2} = \lim\limits_{x\to\infty} \sec^2(1/x) = 1^2 = 1$

45.
$$\lim_{x\to0}\left(\frac{1}{x} - \csc x\right) = \lim_{x\to0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to0} \frac{\sin x - x}{x\sin x}$$
$$\overset{\text{H}}{=} \lim_{x\to0} \frac{\cos x - 1}{x\cos x + \sin x} \overset{\text{H}}{=} \lim_{x\to0} \frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

46. $\lim\limits_{x\to0}(\csc x - \cot x) = \lim\limits_{x\to0}\left(\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}\right) = \lim\limits_{x\to0} \dfrac{1-\cos x}{\sin x} \overset{\text{H}}{=} \lim\limits_{x\to0} \dfrac{\sin x}{\cos x} = 0$

47. We will multiply and divide by the conjugate of the expression to change the form of the expression.

$$\lim_{x\to\infty}\left(\sqrt{x^2+x} - x\right) = \lim_{x\to\infty}\left(\frac{\sqrt{x^2+x}-x}{1}\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right) = \lim_{x\to\infty} \frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}$$
$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}.$$

As an alternate solution, write $\sqrt{x^2+x} - x$ as $\sqrt{x^2+x} - \sqrt{x^2}$, factor out $\sqrt{x^2}$, rewrite as $(\sqrt{1+1/x} - 1)/(1/x)$, and apply l'Hospital's Rule.

48.
$$\lim_{x\to1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to1} \frac{x-1-\ln x}{(x-1)\ln x} \overset{\text{H}}{=} \lim_{x\to1} \frac{1-1/x}{(x-1)(1/x)+\ln x}\cdot\frac{x}{x}$$
$$= \lim_{x\to1} \frac{x-1}{x-1+x\ln x} \overset{\text{H}}{=} \lim_{x\to1} \frac{1}{1+1+\ln x} = \frac{1}{2+0} = \frac{1}{2}$$

49. The limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x.

$$\lim_{x\to\infty}(x - \ln x) = \lim_{x\to\infty} x\left(1 - \frac{\ln x}{x}\right) = \infty \text{ since } \lim_{x\to\infty}\frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

50. As $x \to \infty$, $1/x \to 0$, and $e^{1/x} \to 1$. So the limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x.

$$\lim_{x\to\infty}\left(xe^{1/x} - x\right) = \lim_{x\to\infty} x\left(e^{1/x} - 1\right) = \lim_{x\to\infty}\frac{e^{1/x} - 1}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^{1/x}(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} e^{1/x} = e^0 = 1$$

51. $y = x^{x^2} \quad \Rightarrow \quad \ln y = x^2 \ln x$, so

$$\lim_{x\to0^+}\ln y = \lim_{x\to0^+} x^2\ln x = \lim_{x\to0^+}\frac{\ln x}{1/x^2} \overset{\text{H}}{=} \lim_{x\to0^+}\frac{1/x}{-2/x^3} = \lim_{x\to0^+}\left(-\frac{1}{2}x^2\right) = 0 \quad \Rightarrow$$

$$\lim_{x\to0^+} x^{x^2} = \lim_{x\to0^+} e^{\ln y} = e^0 = 1.$$

52. $y = (\tan 2x)^x \quad \Rightarrow \quad \ln y = x \cdot \ln\tan 2x$, so

$$\lim_{x\to0^+}\ln y = \lim_{x\to0^+} x\cdot\ln\tan 2x = \lim_{x\to0^+}\frac{\ln\tan 2x}{1/x}$$

$$\overset{\text{H}}{=} \lim_{x\to0^+}\frac{(1/\tan 2x)(2\sec^2 2x)}{-1/x^2} = \lim_{x\to0^+}\frac{-2x^2\cos 2x}{\sin 2x\cos^2 2x} = \lim_{x\to0^+}\frac{2x}{\sin 2x}\cdot\lim_{x\to0^+}\frac{-x}{\cos 2x} = 1\cdot 0 = 0 \quad \Rightarrow$$

$$\lim_{x\to0^+}(\tan 2x)^x = \lim_{x\to0^+} e^{\ln y} = e^0 = 1.$$

53. $y = (1-2x)^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x}\ln(1-2x)$, so $\displaystyle\lim_{x\to0}\ln y = \lim_{x\to0}\frac{\ln(1-2x)}{x} \overset{\text{H}}{=} \lim_{x\to0}\frac{-2/(1-2x)}{1} = -2 \quad \Rightarrow$

$\displaystyle\lim_{x\to0}(1-2x)^{1/x} = \lim_{x\to0} e^{\ln y} = e^{-2}.$

54. $y = \left(1+\dfrac{a}{x}\right)^{bx} \quad \Rightarrow \quad \ln y = bx\ln\left(1+\dfrac{a}{x}\right)$, so

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{b\ln(1+a/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{b\left(\dfrac{1}{1+a/x}\right)\left(-\dfrac{a}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty}\frac{ab}{1+a/x} = ab \quad \Rightarrow$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty} e^{\ln y} = e^{ab}.$$

55. $y = \left(1+\dfrac{3}{x}+\dfrac{5}{x^2}\right)^x \quad \Rightarrow \quad \ln y = x\ln\left(1+\dfrac{3}{x}+\dfrac{5}{x^2}\right) \quad \Rightarrow$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln\left(1+\dfrac{3}{x}+\dfrac{5}{x^2}\right)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\left(-\dfrac{3}{x^2}-\dfrac{10}{x^3}\right)\Big/\left(1+\dfrac{3}{x}+\dfrac{5}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty}\frac{3+\dfrac{10}{x}}{1+\dfrac{3}{x}+\dfrac{5}{x^2}} = 3,$$

so $\displaystyle\lim_{x\to\infty}\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^3.$

56. $y = x^{(\ln 2)/(1+\ln x)} \quad \Rightarrow \quad \ln y = \dfrac{\ln 2}{1+\ln x}\ln x \quad \Rightarrow$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{(\ln 2)(\ln x)}{1+\ln x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{(\ln 2)(1/x)}{1/x} = \lim_{x\to\infty}\ln 2 = \ln 2,$$

so $\displaystyle\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)} = \lim_{x\to\infty} e^{\ln y} = e^{\ln 2} = 2.$

57. $y = x^{1/x} \quad \Rightarrow \quad \ln y = (1/x)\ln x \quad \Rightarrow \quad \displaystyle\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{1/x}{1} = 0 \quad \Rightarrow$

$\displaystyle\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$

58. $y = (e^x + x)^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x}\ln(e^x + x)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(e^x + x)}{x} \stackrel{\text{H}}{=} \lim_{x\to\infty} \frac{e^x + 1}{e^x + x} \stackrel{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x + 1} \stackrel{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = 1 \quad \Rightarrow$$

$$\lim_{x\to\infty} (e^x + x)^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^1 = e.$$

59. $y = \left(\dfrac{x}{x+1}\right)^x \quad \Rightarrow \quad \ln y = x\ln\left(\dfrac{x}{x+1}\right) \quad \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right) = \lim_{x\to\infty} \frac{\ln x - \ln(x+1)}{1/x} \stackrel{\text{H}}{=} \lim_{x\to\infty} \frac{1/x - 1/(x+1)}{-1/x^2}$$

$$= \lim_{x\to\infty} \left(-x + \frac{x^2}{x+1}\right) = \lim_{x\to\infty} \frac{-x}{x+1} = -1$$

so $\displaystyle\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^{-1}$

Or: $\displaystyle\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty} \left[\left(\frac{x+1}{x}\right)^{-1}\right]^x = \left[\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x\right]^{-1} = e^{-1}$

60. $y = (\cos 3x)^{5/x} \quad \Rightarrow \quad \ln y = \dfrac{5}{x}\ln(\cos 3x) \quad \Rightarrow \quad \displaystyle\lim_{x\to 0} \ln y = 5\lim_{x\to 0} \frac{\ln(\cos 3x)}{x} \stackrel{\text{H}}{=} 5\lim_{x\to 0} \frac{-3\tan 3x}{1} = 0,$

so $\displaystyle\lim_{x\to 0} (\cos 3x)^{5/x} = e^0 = 1.$

61. $y = (\cos x)^{1/x^2} \quad \Rightarrow \quad \ln y = \dfrac{1}{x^2}\ln\cos x \quad \Rightarrow$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln\cos x}{x^2} \stackrel{\text{H}}{=} \lim_{x\to 0^+} \frac{-\tan x}{2x} \stackrel{\text{H}}{=} \lim_{x\to 0^+} \frac{-\sec^2 x}{2} = -\frac{1}{2} \quad \Rightarrow$$

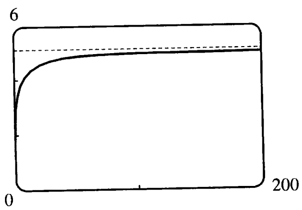
$$\lim_{x\to 0^+} (\cos x)^{1/x^2} = \lim_{x\to 0^+} e^{\ln y} = e^{-1/2} = 1/\sqrt{e}$$

62. $y = \left(\dfrac{2x-3}{2x+5}\right)^{2x+1} \quad \Rightarrow \quad \ln y = (2x+1)\ln\left(\dfrac{2x-3}{2x+5}\right) \quad \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(2x-3) - \ln(2x+5)}{1/(2x+1)} \stackrel{\text{H}}{=} \lim_{x\to\infty} \frac{2/(2x-3) - 2/(2x+5)}{-2/(2x+1)^2} = \lim_{x\to\infty} \frac{-8(2x+1)^2}{(2x-3)(2x+5)}$$

$$= \lim_{x\to\infty} \frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)} = -8 \quad \Rightarrow \quad \lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1} = e^{-8}$$

63.



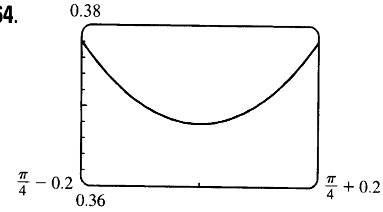
From the graph, it appears that $\displaystyle\lim_{x\to\infty} x\,[\ln(x+5) - \ln x] = 5$.

To prove this, we first note that

$\ln(x+5) - \ln x = \ln\dfrac{x+5}{x} = \ln\left(1 + \dfrac{5}{x}\right) \to \ln 1 = 0$ as $x \to \infty$. Thus,

$$\lim_{x\to\infty} x\,[\ln(x+5) - \ln x] = \lim_{x\to\infty} \frac{\ln(x+5) - \ln x}{1/x} \stackrel{\text{H}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{x+5} - \dfrac{1}{x}}{-1/x^2}$$

$$= \lim_{x\to\infty} \left[\frac{x - (x+5)}{x(x+5)} \cdot \frac{-x^2}{1}\right] = \lim_{x\to\infty} \frac{5x^2}{x^2 + 5x} = 5$$

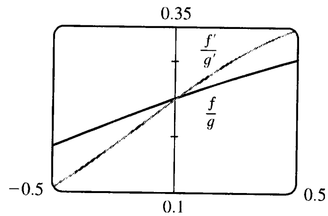
64. 

From the graph, it appears that $\lim\limits_{x\to\pi/4} (\tan x)^{\tan 2x} \approx 0.368$.

The limit has the form 1^∞. Now $y = (\tan x)^{\tan 2x} \Rightarrow$ $\ln y = \tan 2x \ln(\tan x)$, so

$$\lim_{x\to\pi/4} \ln y = \lim_{x\to\pi/4} \frac{\ln(\tan x)}{\cot 2x} \overset{\text{H}}{=} \lim_{x\to\pi/4} \frac{\sec^2 x/\tan x}{-2\csc^2 2x} = \frac{2/1}{-2(1)} = -1$$

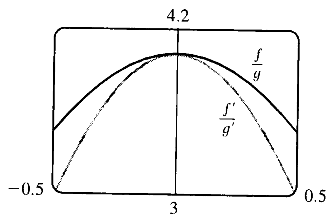
$$\Rightarrow \lim_{x\to\pi/4} (\tan x)^{\tan 2x} = \lim_{x\to\pi/4} e^{\ln y} = e^{-1} = 1/e \approx 0.3679.$$

65. 

From the graph, it appears that

$\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)} = 0.25$. We calculate

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{e^x - 1}{x^3 + 4x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{e^x}{3x^2 + 4} = \frac{1}{4}.$$

66. 

From the graph, it appears that $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)} = 4$.

We calculate

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{2x\sin x}{\sec x - 1} \overset{\text{H}}{=} \lim_{x\to 0} \frac{2(x\cos x + \sin x)}{\sec x\,\tan x}$$

$$\overset{\text{H}}{=} \lim_{x\to 0} \frac{2(-x\sin x + \cos x + \cos x)}{\sec x(\sec^2 x) + \tan x(\sec x\,\tan x)} = \frac{4}{1} = 4$$

67. $\lim\limits_{x\to\infty} \dfrac{e^x}{x^n} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{nx^{n-1}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{n(n-1)x^{n-2}} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{n!} = \infty$

68. $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^p} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{px^{p-1}} = \lim\limits_{x\to\infty} \dfrac{1}{px^p} = 0$ since $p > 0$.

69. First we will find $\lim\limits_{n\to\infty} \left(1 + \dfrac{i}{n}\right)^{nt}$, which is of the form 1^∞. $y = \left(1 + \dfrac{i}{n}\right)^{nt} \Rightarrow \ln y = nt\ln\left(1 + \dfrac{i}{n}\right)$, so

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} nt\ln\left(1 + \frac{i}{n}\right) = t\lim_{n\to\infty} \frac{\ln(1 + i/n)}{1/n} \overset{\text{H}}{=} t\lim_{n\to\infty} \frac{(-i/n^2)}{(1 + i/n)(-1/n^2)} = t\lim_{n\to\infty} \frac{i}{1 + i/n} = ti$$

$\Rightarrow \lim\limits_{n\to\infty} y = e^{it}$. Thus, as $n \to \infty$, $A = A_0\left(1 + \dfrac{i}{n}\right)^{nt} \to A_0 e^{it}$.

70. (a) $\lim\limits_{t\to\infty} v = \lim\limits_{t\to\infty} \dfrac{mg}{c}\left(1 - e^{-ct/m}\right) = \dfrac{mg}{c} \lim\limits_{t\to\infty} \left(1 - e^{-ct/m}\right)$

$= \dfrac{mg}{c}(1 - 0)$ [because $-ct/m \to -\infty$ as $t \to \infty$] $= \dfrac{mg}{c}$,

which is the speed the object approaches as time goes on, the so-called limiting velocity.

(b) $\lim\limits_{m\to\infty} v = \lim\limits_{m\to\infty} \dfrac{mg}{c}\left(1 - e^{-ct/m}\right) = \dfrac{g}{c} \lim\limits_{m\to\infty} \dfrac{1 - e^{-ct/m}}{1/m} \overset{\text{H}}{=} \dfrac{g}{c} \lim\limits_{m\to\infty} \dfrac{-e^{-ct/m}(ct/m^2)}{-1/m^2}$

$= \dfrac{g}{c}(ct) \lim\limits_{m\to\infty} e^{-ct/m} = gt(1)$ [because $-ct/m \to 0$ as $m \to \infty$] $= gt$.

The speed of a very heavy falling object is approximately proportional to the elapsed time t, provided it can fall for time t in an environment where the given model continues to hold. [If t is too large, the object may hit the ground in less than time t, or it may have to start falling too high above the earth, where there is almost no air.].

71. We see that both numerator and denominator approach 0, so we can use l'Hospital's Rule:

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{aax}}{a-\sqrt[4]{ax^3}} \overset{\text{H}}{=} \lim_{x\to a}\frac{\frac{1}{2}\left(2a^3x-x^4\right)^{-1/2}\left(2a^3-4x^3\right)-a\left(\frac{1}{3}\right)(aax)^{-2/3}a^2}{-\frac{1}{4}\left(ax^3\right)^{-3/4}\left(3ax^2\right)}$$

$$=\frac{\frac{1}{2}\left(2a^3a-a^4\right)^{-1/2}\left(2a^3-4a^3\right)-\frac{1}{3}a^3\left(a^2a\right)^{-2/3}}{-\frac{1}{4}\left(aa^3\right)^{-3/4}\left(3aa^2\right)}$$

$$=\frac{\left(a^4\right)^{-1/2}\left(-a^3\right)-\frac{1}{3}a^3\left(a^3\right)^{-2/3}}{-\frac{3}{4}a^3\left(a^4\right)^{-3/4}}=\frac{-a-\frac{1}{3}a}{-\frac{3}{4}}=\frac{4}{3}\left(\frac{4}{3}a\right)=\frac{16}{9}a$$

72. Let the radius of the circle be r. We see that $A(\theta)$ is the area of the whole figure (a sector of the circle with radius 1), minus the area of $\triangle OPR$. But the area of the sector of the circle is $\frac{1}{2}r^2\theta$ (see Reference Page 1), and the area of the triangle is $\frac{1}{2}r\,|PQ|=\frac{1}{2}r(r\sin\theta)=\frac{1}{2}r^2\sin\theta$. So we have

$A(\theta)=\frac{1}{2}r^2\theta-\frac{1}{2}r^2\sin\theta=\frac{1}{2}r^2(\theta-\sin\theta)$. Now by elementary trigonometry,

$B(\theta)=\frac{1}{2}\,|QR|\,|PQ|=\frac{1}{2}(r-|OQ|)\,|PQ|=\frac{1}{2}(r-r\cos\theta)(r\sin\theta)=\frac{1}{2}r^2(1-\cos\theta)\sin\theta$.

So the limit we want is

$$\lim_{\theta\to0^+}\frac{A(\theta)}{B(\theta)}=\lim_{\theta\to0^+}\frac{\frac{1}{2}r^2(\theta-\sin\theta)}{\frac{1}{2}r^2(1-\cos\theta)\sin\theta}\overset{\text{H}}{=}\lim_{\theta\to0^+}\frac{1-\cos\theta}{(1-\cos\theta)\cos\theta+\sin\theta\,(\sin\theta)}$$

$$=\lim_{\theta\to0^+}\frac{1-\cos\theta}{\cos\theta-\cos^2\theta+\sin^2\theta}\overset{\text{H}}{=}\lim_{\theta\to0^+}\frac{\sin\theta}{-\sin\theta-2\cos\theta\,(-\sin\theta)+2\sin\theta\,(\cos\theta)}$$

$$=\lim_{\theta\to0^+}\frac{\sin\theta}{-\sin\theta+4\sin\theta\,\cos\theta}=\lim_{\theta\to0^+}\frac{1}{-1+4\cos\theta}=\frac{1}{-1+4\cos0}=\frac{1}{3}$$

73. Since $f(2)=0$, the given limit has the form $\frac{0}{0}$.

$$\lim_{x\to0}\frac{f(2+3x)+f(2+5x)}{x}\overset{\text{H}}{=}\lim_{x\to0}\frac{f'(2+3x)\cdot3+f'(2+5x)\cdot5}{1}=f'(2)\cdot3+f'(2)\cdot5=8f'(2)=8\cdot7=56$$

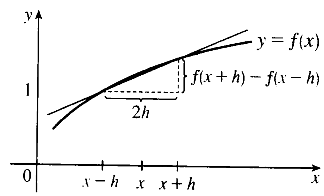
74. $L=\lim\limits_{x\to0}\left(\dfrac{\sin2x}{x^3}+a+\dfrac{b}{x^2}\right)=\lim\limits_{x\to0}\dfrac{\sin2x+ax^3+bx}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to0}\dfrac{2\cos2x+3ax^2+b}{3x^2}$. As $x\to0$, $3x^2\to0$, and $\left(2\cos2x+3ax^2+b\right)\to b+2$, so the last limit exists only if $b+2=0$, that is, $b=-2$. Thus,

$\lim\limits_{x\to0}\dfrac{2\cos2x+3ax^2-2}{3x^2}\overset{\text{H}}{=}\lim\limits_{x\to0}\dfrac{-4\sin2x+6ax}{6x}\overset{\text{H}}{=}\lim\limits_{x\to0}\dfrac{-8\cos2x+6a}{6}=\dfrac{6a-8}{6}$, which is equal to 0 if and only if $a=\frac{4}{3}$. Hence, $L=0$ if and only if $b=-2$ and $a=\frac{4}{3}$.

75. Since $\lim_{h\to 0}[f(x+h)-f(x-h)] = f(x)-f(x) = 0$ (f is differentiable and hence continuous) and $\lim_{h\to 0} 2h = 0$, we use l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} \overset{\text{H}}{=} \lim_{h\to 0}\frac{f'(x+h)(1)-f'(x-h)(-1)}{2} = \frac{f'(x)+f'(x)}{2} = \frac{2f'(x)}{2} = f'(x)$$

$\dfrac{f(x+h)-f(x-h)}{2h}$ is the slope of the secant line between $(x-h, f(x-h))$ and $(x+h, f(x+h))$. As $h\to 0$, this line gets closer to the tangent line and its slope approaches $f'(x)$.



76. Since $\lim_{h\to 0}[f(x+h)-2f(x)+f(x-h)] = f(x)-2f(x)+f(x) = 0$ (f is differentiable and hence continuous) and $\lim_{h\to 0} h^2 = 0$, we can apply l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} \overset{\text{H}}{=} \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} = f''(x)$$

At the last step, we have applied the result of Exercise 75 to $f'(x)$.

77. (a) We show that $\lim_{x\to 0}\dfrac{f(x)}{x^n} = 0$ for every integer $n \geq 0$. Let $y = \dfrac{1}{x^2}$. Then

$$\lim_{x\to 0}\frac{f(x)}{x^{2n}} = \lim_{x\to 0}\frac{e^{-1/x^2}}{(x^2)^n} = \lim_{y\to\infty}\frac{y^n}{e^y} \overset{\text{H}}{=} \lim_{y\to\infty}\frac{ny^{n-1}}{e^y} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim_{y\to\infty}\frac{n!}{e^y} = 0 \Rightarrow$$

$$\lim_{x\to 0}\frac{f(x)}{x^n} = \lim_{x\to 0} x^n\frac{f(x)}{x^{2n}} = \lim_{x\to 0}x^n \lim_{x\to 0}\frac{f(x)}{x^{2n}} = 0. \text{ Thus, } f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)}{x} = 0.$$

(b) Using the Chain Rule and the Quotient Rule we see that $f^{(n)}(x)$ exists for $x \neq 0$. In fact, we prove by induction that for each $n \geq 0$, there is a polynomial p_n and a non-negative integer k_n with $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$. This is true for $n = 0$; suppose it is true for the nth derivative. Then $f'(x) = f(x)(2/x^3)$, so

$$\begin{aligned} f^{(n+1)}(x) &= \left[x^{k_n}\left[p_n'(x)f(x)+p_n(x)f'(x)\right] - k_n x^{k_n-1}p_n(x)f(x)\right]x^{-2k_n} \\ &= \left[x^{k_n}p_n'(x)+p_n(x)(2/x^3) - k_n x^{k_n-1}p_n(x)\right]f(x)x^{-2k_n} \\ &= \left[x^{k_n+3}p_n'(x)+2p_n(x) - k_n x^{k_n+2}p_n(x)\right]f(x)x^{-(2k_n+3)} \end{aligned}$$

which has the desired form.

Now we show by induction that $f^{(n)}(0) = 0$ for all n. By part (a), $f'(0) = 0$. Suppose that $f^{(n)}(0) = 0$. Then

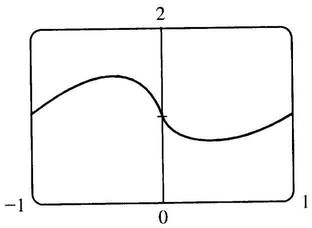
$$\begin{aligned} f^{(n+1)}(0) &= \lim_{x\to 0}\frac{f^{(n)}(x)-f^{(n)}(0)}{x-0} = \lim_{x\to 0}\frac{f^{(n)}(x)}{x} = \lim_{x\to 0}\frac{p_n(x)f(x)/x^{k_n}}{x} = \lim_{x\to 0}\frac{p_n(x)f(x)}{x^{k_n+1}} \\ &= \lim_{x\to 0}p_n(x)\lim_{x\to 0}\frac{f(x)}{x^{k_n+1}} = p_n(0)\cdot 0 = 0 \end{aligned}$$

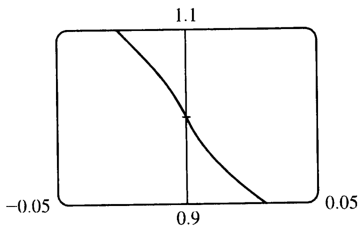
78. (a) For f to be continuous, we need $\lim_{x\to 0} f(x) = f(0) = 1$. We note that for $x \neq 0$, $\ln f(x) = \ln |x|^x = x \ln |x|$.

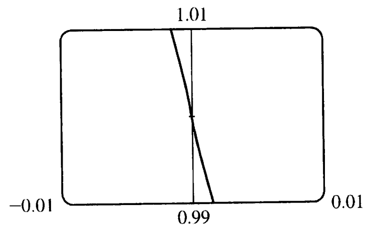
So $\lim_{x\to 0} \ln f(x) = \lim_{x\to 0} x \ln |x| = \lim_{x\to 0} \dfrac{\ln |x|}{1/x} \overset{\text{H}}{=} \lim_{x\to 0} \dfrac{1/x}{-1/x^2} = 0$. Therefore,

$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\ln f(x)} = e^0 = 1$. So f is continuous at 0.

(b) From the graphs, it appears that f is differentiable at 0.







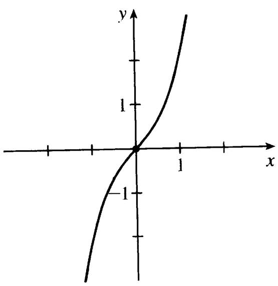
(c) To find f', we use logarithmic differentiation: $\ln f(x) = x \ln |x| \;\Rightarrow\; \dfrac{f'(x)}{f(x)} = x\left(\dfrac{1}{x}\right) + \ln |x| \;\Rightarrow$

$f'(x) = f(x)(1 + \ln |x|) = |x|^x (1 + \ln |x|)$, $x \neq 0$. Now $f'(x) \to -\infty$ as $x \to 0$ [since $|x|^x \to 1$ and $(1 + \ln |x|) \to -\infty$], so the curve has a vertical tangent at $(0, 1)$ and is therefore not differentiable there. The fact cannot be seen in the graphs in part (b) because $\ln |x| \to -\infty$ very slowly as $x \to 0$.

4.5 Summary of Curve Sketching

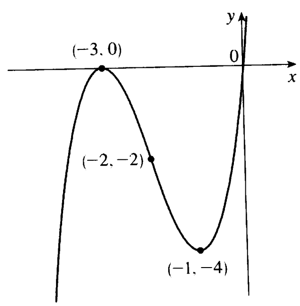
1. $y = f(x) = x^3 + x = x(x^2 + 1)$ **A.** f is a polynomial, so $D = \mathbb{R}$. **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. **D.** f is a polynomial, so there is no asymptote. **E.** $f'(x) = 3x^2 + 1 > 0$, so f is increasing on $(-\infty, \infty)$. **F.** There is no critical number and hence, no local maximum or minimum value. **G.** $f''(x) = 6x > 0$ on $(0, \infty)$ and $f''(x) < 0$ on $(-\infty, 0)$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. Since the concavity changes at $x = 0$, there is an inflection point at $(0, 0)$.

H.



2. $y = f(x) = x^3 + 6x^2 + 9x = x(x + 3)^2$ **A.** $D = \mathbb{R}$ **B.** x-intercepts are -3 and 0, y-intercept $= 0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 3x^2 + 12x + 9 = 3(x + 1)(x + 3) < 0 \;\Leftrightarrow\; -3 < x < -1$, so f is decreasing on $(-3, -1)$ and increasing on $(-\infty, -3)$ and $(-1, \infty)$. **F.** Local maximum value $f(-3) = 0$, local minimum value $f(-1) = -4$ **G.** $f''(x) = 6x + 12 = 6(x + 2) > 0 \;\Leftrightarrow\; x > -2$, so f is CU on $(-2, \infty)$ and CD on $(-\infty, -2)$. IP at $(-2, -2)$

H.



3. $y = f(x) = 2 - 15x + 9x^2 - x^3 = -(x-2)(x^2 - 7x + 1)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$; x-intercepts: $f(x) = 0 \Rightarrow x = 2$ or (by the quadratic formula) $x = \frac{7 \pm \sqrt{45}}{2} \approx 0.15, 6.85$

C. No symmetry **D.** No asymptote

E. $f'(x) = -15 + 18x - 3x^2 = -3(x^2 - 6x + 5)$

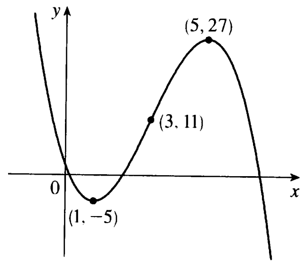
$= -3(x-1)(x-5) > 0 \Leftrightarrow 1 < x < 5$

so f is increasing on $(1, 5)$ and decreasing on $(-\infty, 1)$ and $(5, \infty)$.

F. Local maximum value $f(5) = 27$, local minimum value $f(1) = -5$

G. $f''(x) = 18 - 6x = -6(x-3) > 0 \Leftrightarrow x < 3$, so f is CU on $(-\infty, 3)$ and CD on $(3, \infty)$. IP at $(3, 11)$

H.



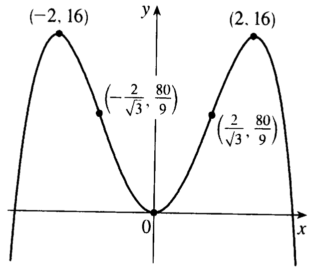
4. $y = f(x) = 8x^2 - x^4 = x^2(8 - x^2)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 0, \pm 2\sqrt{2}$ $(\approx \pm 2.83)$ **C.** $f(-x) = f(x)$, so f is even and symmetric about the y-axis. **D.** No asymptote

E. $f'(x) = 16x - 4x^3 = 4x(4 - x^2) = 4x(2+x)(2-x) > 0 \Leftrightarrow$ $x < -2$ or $0 < x < 2$, so f is increasing on $(-\infty, -2)$ and $(0, 2)$ and decreasing on $(-2, 0)$ and $(2, \infty)$. **F.** Local maximum value $f(\pm 2) = 16$, local minimum value $f(0) = 0$

G. $f''(x) = 16 - 12x^2 = 4(4 - 3x^2) = 0 \Leftrightarrow x = \pm\frac{2}{\sqrt{3}}$.

$f''(x) > 0 \Leftrightarrow -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$, so f is CU on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and CD on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$. IP at $\left(\pm\frac{2}{\sqrt{3}}, \frac{80}{9}\right)$

H.

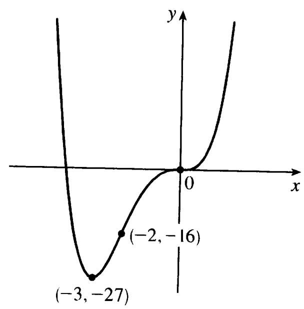


5. $y = f(x) = x^4 + 4x^3 = x^3(x+4)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow x = -4, 0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 4x^3 + 12x^2 = 4x^2(x+3) > 0 \Leftrightarrow$ $x > -3$, so f is increasing on $(-3, \infty)$ and decreasing on $(-\infty, -3)$.

F. Local minimum value $f(-3) = -27$, no local maximum

G. $f''(x) = 12x^2 + 24x = 12x(x+2) < 0 \Leftrightarrow -2 < x < 0$, so f is CD on $(-2, 0)$ and CU on $(-\infty, -2)$ and $(0, \infty)$. IP at $(0, 0)$ and $(-2, -16)$

H.



6. $y = f(x) = x(x+2)^3$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow x = -2, 0$ **C.** No symmetry **D.** No asymptote

E. $f'(x) = 3x(x+2)^2 + (x+2)^3 = (x+2)^2\,[3x + (x+2)] = (x+2)^2(4x+2)$. $f'(x) > 0 \Leftrightarrow x > -\frac{1}{2}$, and $f'(x) < 0 \Leftrightarrow x < -2$ or $-2 < x < -\frac{1}{2}$, so f is increasing on $\left(-\frac{1}{2}, \infty\right)$ and decreasing on $(-\infty, -2)$

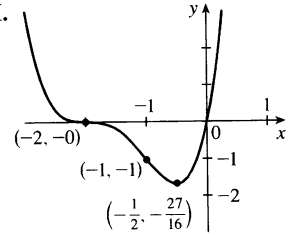
and $\left(-2, -\frac{1}{2}\right)$. [Hence f is decreasing on $\left(-\infty, -\frac{1}{2}\right)$ by the analogue of Exercise 4.3.65 for decreasing functions.]

F. Local minimum value $f\left(-\frac{1}{2}\right) = -\frac{27}{16}$, no local maximum

G. $f''(x) = (x+2)^2(4) + (4x+2)(2)(x+2)$
$= 2(x+2)[(x+2)(2) + 4x + 2]$
$= 2(x+2)(6x+6) = 12(x+1)(x+2)$

$f''(x) < 0 \quad\Leftrightarrow\quad -2 < x < -1$, so f is CD on $(-2, -1)$ and CU on $(-\infty, -2)$ and $(-1, \infty)$. IP at $(-2, 0)$ and $(-1, -1)$

H.



7. $y = f(x) = 2x^5 - 5x^2 + 1$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$ **C.** No symmetry **D.** No asymptote

E. $f'(x) = 10x^4 - 10x = 10x(x^3 - 1) = 10x(x-1)(x^2+x+1)$, so $f'(x) < 0 \quad\Leftrightarrow\quad 0 < x < 1$ and $f'(x) > 0 \quad\Leftrightarrow\quad x < 0$ or $x > 1$. Thus, f is increasing on $(-\infty, 0)$ and $(1, \infty)$ and decreasing on $(0, 1)$.

F. Local maximum value $f(0) = 1$, local minimum value $f(1) = -2$

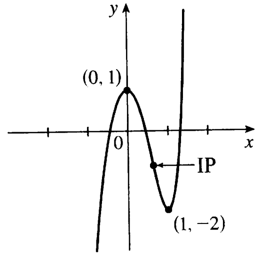
G. $f''(x) = 40x^3 - 10 = 10(4x^3 - 1)$ so $f''(x) = 0 \quad\Leftrightarrow\quad x = 1/\sqrt[3]{4}$.

$f''(x) > 0 \quad\Leftrightarrow\quad x > 1/\sqrt[3]{4}$ and $f''(x) < 0 \quad\Leftrightarrow\quad x < 1/\sqrt[3]{4}$,

so f is CD on $\left(-\infty, 1/\sqrt[3]{4}\right)$ and CU on $\left(1/\sqrt[3]{4}, \infty\right)$.

IP at $\left(\frac{1}{\sqrt[3]{4}}, 1 - \frac{9}{2\left(\sqrt[3]{4}\right)^2}\right) \approx (0.630, -0.786)$

H.



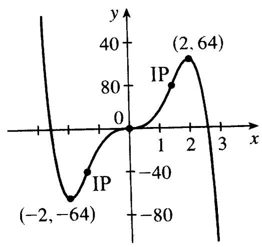
8. $y = f(x) = 20x^3 - 3x^5$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad\Leftrightarrow\quad -3x^3\left(x^2 - \frac{20}{3}\right) = 0 \quad\Leftrightarrow\quad x = 0$ or $\pm\sqrt{20/3} \approx \pm 2.582$ **C.** $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. **D.** No asymptote

E. $f'(x) = 60x^2 - 15x^4 = -15x^2(x^2 - 4) = -15x^2(x+2)(x-2)$, so $f'(x) > 0 \quad\Leftrightarrow\quad -2 < x < 0$ or $0 < x < 2$ and $f'(x) < 0 \quad\Leftrightarrow\quad x < -2$ or $x > 2$. Thus, f is increasing on $(-2, 0)$ and $(0, 2)$ [hence on $(-2, 2)$ by Exercise 4.3.65] and f is decreasing on $(-\infty, -2)$ and $(2, \infty)$.

F. Local minimum value $f(-2) = -64$, local maximum value $f(2) = 64$ **G.** $f''(x) = 120x - 60x^3 = -60x(x^2 - 2)$.

$f''(x) > 0 \quad\Leftrightarrow\quad x < -\sqrt{2}$ or $0 < x < \sqrt{2}$; $f''(x) < 0 \quad\Leftrightarrow\quad -\sqrt{2} < x < 0$ or $x > \sqrt{2}$. Thus, f is CU on $\left(-\infty, -\sqrt{2}\right)$ and $\left(0, \sqrt{2}\right)$, and f is CD on $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, \infty\right)$. IP at $\left(-\sqrt{2}, -28\sqrt{2}\right) \approx (-1.414, -39.598)$, $(0, 0)$, and $\left(\sqrt{2}, 28\sqrt{2}\right)$

H.



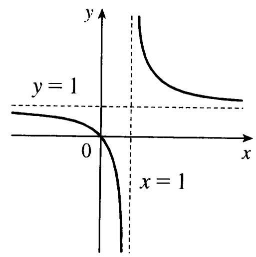
9. $y = f(x) = x/(x-1)$ **A.** $D = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty} \dfrac{x}{x-1} = 1$, so $y = 1$ is a HA.

$\lim\limits_{x\to 1^-} \dfrac{x}{x-1} = -\infty$, $\lim\limits_{x\to 1^+} \dfrac{x}{x-1} = \infty$, so $x = 1$ is a VA.

E. $f'(x) = \dfrac{(x-1)-x}{(x-1)^2} = \dfrac{-1}{(x-1)^2} < 0$ for $x \neq 1$, so f is decreasing on $(-\infty, 1)$ and $(1, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{2}{(x-1)^3} > 0 \iff x > 1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, 1)$. No IP

H.



10. $y = x/(x-1)^2$ **A.** $D = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$

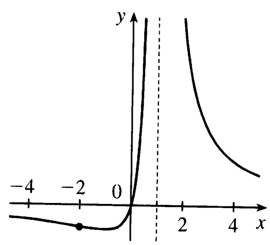
C. No symmetry **D.** $\lim\limits_{x\to\pm\infty} \dfrac{x}{(x-1)^2} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to 1} \dfrac{x}{(x-1)^2} = \infty$, so $x = 1$ is a VA.

E. $f'(x) = \dfrac{(x-1)^2(1) - x(2)(x-1)}{(x-1)^4} = \dfrac{-x-1}{(x-1)^3}$. This is negative on $(-\infty, -1)$ and $(1, \infty)$ and positive on $(-1, 1)$, so $f(x)$ is decreasing on $(-\infty, -1)$ and $(1, \infty)$ and increasing on $(-1, 1)$.

F. Local minimum value $f(-1) = -\frac{1}{4}$, no local maximum.

G. $f''(x) = \dfrac{(x-1)^3(-1) + (x+1)(3)(x-1)^2}{(x-1)^6} = \dfrac{2(x+2)}{(x-1)^4}$. This is negative on $(-\infty, -2)$, and positive on $(-2, 1)$ and $(1, \infty)$. So f is CD on $(-\infty, -2)$ and CU on $(-2, 1)$ and $(1, \infty)$. IP at $\left(-2, -\frac{2}{9}\right)$

H.



11. $y = f(x) = 1/(x^2 - 9)$ **A.** $D = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

B. y-intercept $= f(0) = -\frac{1}{9}$, no x-intercept **C.** $f(-x) = f(x) \Rightarrow f$ is even; the curve is symmetric about the y-axis. **D.** $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2-9} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to 3^-} \dfrac{1}{x^2-9} = -\infty$, $\lim\limits_{x\to 3^+} \dfrac{1}{x^2-9} = \infty$,

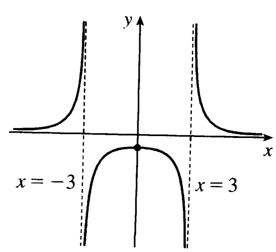
$\lim\limits_{x\to -3^-} \dfrac{1}{x^2-9} = \infty$, $\lim\limits_{x\to -3^+} \dfrac{1}{x^2-9} = -\infty$, so $x = 3$ and $x = -3$ are VA.

E. $f'(x) = -\dfrac{2x}{(x^2-9)^2} > 0 \iff x < 0$ $(x \neq -3)$ so f is increasing on $(-\infty, -3)$ and $(-3, 0)$ and decreasing on $(0, 3)$ and $(3, \infty)$.

F. Local maximum value $f(0) = -\frac{1}{9}$.

G. $y'' = \dfrac{-2(x^2-9)^2 + (2x)2(x^2-9)(2x)}{(x^2-9)^4} = \dfrac{6(x^2+3)}{(x^2-9)^3} > 0 \iff x^2 > 9 \iff x > 3$ or $x < -3$, so f is CU on $(-\infty, -3)$ and $(3, \infty)$ and CD on $(-3, 3)$. No IP

H.



12. $y = f(x) = x/(x^2 - 9)$ **A.** $D = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$. **C.** $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin.

D. $\lim\limits_{x \to \pm\infty} \dfrac{x}{x^2 - 9} = 0$, so $y = 0$ is a HA. $\lim\limits_{x \to 3^+} \dfrac{x}{x^2 - 9} = \infty$, $\lim\limits_{x \to 3^-} \dfrac{x}{x^2 - 9} = -\infty$,

$\lim\limits_{x \to -3^+} \dfrac{x}{x^2 - 9} = \infty$, $\lim\limits_{x \to -3^-} \dfrac{x}{x^2 - 9} = -\infty$, so $x = 3$ and $x = -3$ are VA.

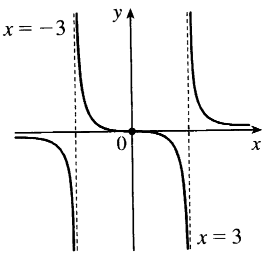
E. $f'(x) = \dfrac{(x^2 - 9) - x(2x)}{(x^2 - 9)^2} = -\dfrac{x^2 + 9}{(x^2 - 9)^2} < 0$ $(x \neq \pm 3)$ so f is decreasing on $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$. **F.** No extreme values

G. $f''(x) = -\dfrac{2x(x^2 - 9)^2 - (x^2 + 9) \cdot 2(x^2 - 9)(2x)}{(x^2 - 9)^4}$

$= \dfrac{2x(x^2 + 27)}{(x^2 - 9)^3} > 0$ when $-3 < x < 0$ or $x > 3$,

so f is CU on $(-3, 0)$ and $(3, \infty)$; CD on $(-\infty, -3)$ and $(0, 3)$. IP at $(0, 0)$

H.



13. $y = f(x) = x/(x^2 + 9)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \Leftrightarrow x = 0$

C. $f(-x) = -f(x)$, so f is odd and the curve is symmetric about the origin. **D.** $\lim\limits_{x \to \pm\infty} [x/(x^2 + 9)] = 0$, so $y = 0$ is a HA; no VA **E.** $f'(x) = \dfrac{(x^2 + 9)(1) - x(2x)}{(x^2 + 9)^2} = \dfrac{9 - x^2}{(x^2 + 9)^2} = \dfrac{(3 + x)(3 - x)}{(x^2 + 9)^2} > 0 \Leftrightarrow$

$-3 < x < 3$, so f is increasing on $(-3, 3)$ and decreasing on $(-\infty, -3)$ and $(3, \infty)$.

F. Local minimum value $f(-3) = -\frac{1}{6}$, local maximum value $f(3) = \frac{1}{6}$

G. $f''(x) = \dfrac{(x^2 + 9)^2(-2x) - (9 - x^2) \cdot 2(x^2 + 9)(2x)}{[(x^2 + 9)^2]^2}$

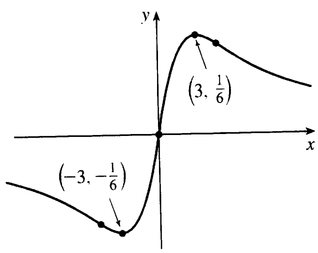
$= \dfrac{(2x)(x^2 + 9)[-(x^2 + 9) - 2(9 - x^2)]}{(x^2 + 9)^4}$

$= \dfrac{2x(x^2 - 27)}{(x^2 + 9)^3} = 0 \quad \Leftrightarrow \quad x = 0, \pm\sqrt{27} = \pm 3\sqrt{3}$

$f''(x) > 0 \quad \Leftrightarrow \quad -3\sqrt{3} < x < 0$ or $x > 3\sqrt{3}$, so f is CU on $(-3\sqrt{3}, 0)$ and $(3\sqrt{3}, \infty)$, and CD on $(-\infty, -3\sqrt{3})$ and $(0, 3\sqrt{3})$.

There are three inflection points: $(0, 0)$ and $(\pm 3\sqrt{3}, \pm\frac{1}{12}\sqrt{3})$.

H.



14. $y = f(x) = x^2/(x^2 + 9)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \Leftrightarrow x = 0$

C. $f(-x) = f(x)$, so f is even and symmetric about the y-axis. **D.** $\lim\limits_{x \to \pm\infty} [x^2/(x^2 + 9)] = 1$, so $y = 1$

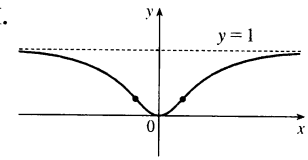
is a HA; no VA **E.** $f'(x) = \dfrac{(x^2+9)(2x) - x^2(2x)}{(x^2+9)^2} = \dfrac{18x}{(x^2+9)^2} > 0 \Leftrightarrow x > 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. **F.** Local minimum value $f(0) = 0$; no local maximum

$$\textbf{G. } f''(x) = \frac{(x^2+9)^2(18) - 18x \cdot 2(x^2+9)\cdot 2x}{[(x^2+9)^2]^2} = \frac{18(x^2+9)[(x^2+9) - 4x^2]}{(x^2+9)^4} = \frac{18(9-3x^2)}{(x^2+9)^3}$$
$$= \frac{-54(x+\sqrt{3})(x-\sqrt{3})}{(x^2+9)^3} > 0 \Leftrightarrow -\sqrt{3} < x < \sqrt{3}$$

so f is CU on $(-\sqrt{3}, \sqrt{3})$ and CD on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$.

There are two inflection points: $(\pm\sqrt{3}, \frac{1}{4})$.

H.



15. $y = f(x) = \dfrac{x-1}{x^2}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No y-intercept; x-intercept: $f(x) = 0 \Leftrightarrow x = 1$ **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty} \dfrac{x-1}{x^2} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to 0} \dfrac{x-1}{x^2} = -\infty$, so $x = 0$ is a VA.

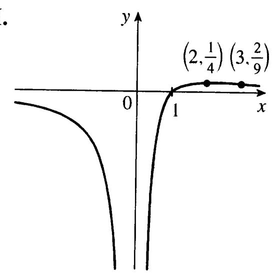
E. $f'(x) = \dfrac{x^2 \cdot 1 - (x-1)\cdot 2x}{(x^2)^2} = \dfrac{-x^2+2x}{x^4} = \dfrac{-(x-2)}{x^3}$, so $f'(x) > 0 \Leftrightarrow 0 < x < 2$ and $f'(x) < 0 \Leftrightarrow x < 0$ or $x > 2$. Thus, f is increasing on $(0, 2)$ and decreasing on $(-\infty, 0)$ and $(2, \infty)$.

F. No local minimum, local maximum value $f(2) = \frac{1}{4}$.

G. $f''(x) = \dfrac{x^3\cdot(-1) - [-(x-2)]\cdot 3x^2}{(x^3)^2} = \dfrac{2x^3 - 6x^2}{x^6} = \dfrac{2(x-3)}{x^4}$.

$f''(x)$ is negative on $(-\infty, 0)$ and $(0, 3)$ and positive on $(3, \infty)$, so f is CD on $(-\infty, 0)$ and $(0, 3)$ and CU on $(3, \infty)$. IP at $(3, \frac{2}{9})$

H.



16. $y = f(x) = \dfrac{x^2-2}{x^4}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No y-intercept; x-intercepts: $f(x) = 0 \Leftrightarrow x = \pm\sqrt{2}$ **C.** $f(-x) = f(x)$, so f is even; the curve is symmetric about the y-axis.

D. $\lim\limits_{x\to\pm\infty} \dfrac{x^2-2}{x^4} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to 0} \dfrac{x^2-2}{x^4} = -\infty$, so $x = 0$ is a VA.

E. $f'(x) = \dfrac{x^4\cdot 2x - (x^2-2)(4x^3)}{(x^4)^2} = \dfrac{-2x^5+8x^3}{x^8} = \dfrac{-2(x^2-4)}{x^5} = \dfrac{-2(x+2)(x-2)}{x^5}$.

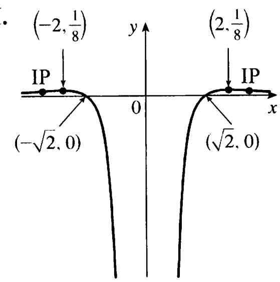
$f'(x)$ is negative on $(-2, 0)$ and $(2, \infty)$ and positive on $(-\infty, -2)$ and $(0, 2)$, so f is decreasing on $(-2, 0)$ and $(2, \infty)$ and increasing on $(-\infty, -2)$ and $(0, 2)$. **F.** Local maximum value $f(\pm 2) = \frac{1}{8}$, no local minimum.

G. $f''(x) = \dfrac{x^5 \cdot (-4x) + 2(x^2-4)\cdot 5x^4}{(x^5)^2} = \dfrac{2x^4[-2x^2+5(x^2-4)]}{x^{10}} = \dfrac{2(3x^2-20)}{x^6}$

$f''(x)$ is positive on $\left(-\infty, -\sqrt{\frac{20}{3}}\right)$ and $\left(\sqrt{\frac{20}{3}}, \infty\right)$ and negative on $\left(-\sqrt{\frac{20}{3}}, 0\right)$ and $\left(0, \sqrt{\frac{20}{3}}\right)$, so f is CU on $\left(-\infty, -\sqrt{\frac{20}{3}}\right)$ and $\left(\sqrt{\frac{20}{3}}, \infty\right)$ and CD on $\left(-\sqrt{\frac{20}{3}}, 0\right)$ and $\left(0, \sqrt{\frac{20}{3}}\right)$.

IP at $\left(\pm\sqrt{\frac{20}{3}}, \frac{21}{200}\right) \approx (\pm 2.5820, 0.105)$

H.



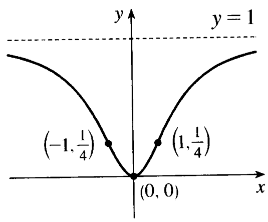
17. $y = f(x) = \dfrac{x^2}{x^2+3} = \dfrac{(x^2+3)-3}{x^2+3} = 1 - \dfrac{3}{x^2+3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow x = 0$ **C.** $f(-x) = f(x)$, so f is even; the graph is symmetric about the y-axis.

D. $\displaystyle\lim_{x\to\pm\infty} \frac{x^2}{x^2+3} = 1$, so $y = 1$ is a HA. No VA. **E.** Using the Reciprocal Rule,

$f'(x) = -3 \cdot \dfrac{-2x}{(x^2+3)^2} = \dfrac{6x}{(x^2+3)^2}$. $f'(x) > 0 \Leftrightarrow x > 0$ and $f'(x) < 0 \Leftrightarrow x < 0$, so f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. **F.** Local minimum value $f(0) = 0$, no local maximum.

G.
$$\begin{aligned} f''(x) &= \frac{(x^2+3)^2 \cdot 6 - 6x \cdot 2(x^2+3)\cdot 2x}{[(x^2+3)^2]^2} \\ &= \frac{6(x^2+3)[(x^2+3)-4x^2]}{(x^2+3)^4} \\ &= \frac{6(3-3x^2)}{(x^2+3)^3} = \frac{-18(x+1)(x-1)}{(x^2+3)^3} \end{aligned}$$

H.



$f''(x)$ is negative on $(-\infty, -1)$ and $(1, \infty)$ and positive on $(-1, 1)$, so f is CD on $(-\infty, -1)$ and $(1, \infty)$ and CU on $(-1, 1)$. IP at $\left(\pm 1, \frac{1}{4}\right)$

18. $y = f(x) = \dfrac{x^3-1}{x^3+1}$ **A.** $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ **B.** x-intercept $= 1$,

y-intercept $= f(0) = -1$ **C.** No symmetry **D.** $\displaystyle\lim_{x\to\pm\infty} \frac{x^3-1}{x^3+1} = \lim_{x\to\pm\infty} \frac{1-1/x^3}{1+1/x^3} = 1$, so $y = 1$ is a HA.

$\displaystyle\lim_{x\to -1^-} \frac{x^3-1}{x^3+1} = \infty$ and $\displaystyle\lim_{x\to -1^+} \frac{x^3-1}{x^3+1} = -\infty$, so $x = -1$ is a VA.

E. $f'(x) = \dfrac{(x^3+1)(3x^2) - (x^3-1)(3x^2)}{(x^3+1)^2} = \dfrac{6x^2}{(x^3+1)^2} > 0$ $(x \neq -1)$ so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values

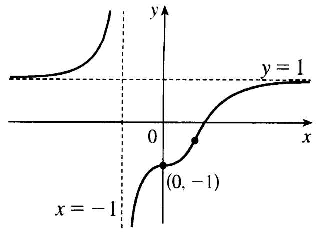
G. $y'' = \dfrac{12x(x^3+1)^2 - 6x^2 \cdot 2(x^3+1)\cdot 3x^2}{(x^3+1)^4}$

$= \dfrac{12x(1-2x^3)}{(x^3+1)^3} > 0 \quad \Leftrightarrow \quad x < -1 \text{ or } 0 < x < \dfrac{1}{\sqrt[3]{2}}$,

so f is CU on $(-\infty, -1)$ and $\left(0, \frac{1}{\sqrt[3]{2}}\right)$ and CD on $(-1, 0)$ and $\left(\frac{1}{\sqrt[3]{2}}, \infty\right)$.

IP at $(0, -1)$, $\left(\frac{1}{\sqrt[3]{2}}, -\frac{1}{3}\right)$

H.



19. $y = f(x) = x\sqrt{5-x}$ **A.** The domain is $\{x \mid 5 - x \geq 0\} = (-\infty, 5]$ **B.** y-intercept: $f(0) = 0$;

x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x = 0, 5$ **C.** No symmetry **D.** No asymptote

E. $f'(x) = x \cdot \frac{1}{2}(5-x)^{-1/2}(-1) + (5-x)^{1/2} \cdot 1 = \frac{1}{2}(5-x)^{-1/2}[-x + 2(5-x)] = \dfrac{10-3x}{2\sqrt{5-x}} > 0 \quad \Leftrightarrow$

$x < \frac{10}{3}$, so f is increasing on $\left(-\infty, \frac{10}{3}\right)$ and decreasing on $\left(\frac{10}{3}, 5\right)$.

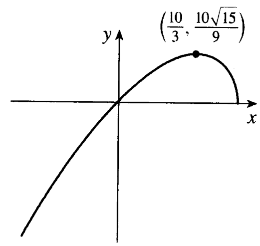
F. Local maximum value $f\left(\frac{10}{3}\right) = \frac{10}{9}\sqrt{15} \approx 4.3$; no local minimum

G. $f''(x) = \dfrac{2(5-x)^{1/2}(-3) - (10-3x)\cdot 2\left(\frac{1}{2}\right)(5-x)^{-1/2}(-1)}{\left(2\sqrt{5-x}\right)^2}$

$= \dfrac{(5-x)^{-1/2}[-6(5-x) + (10-3x)]}{4(5-x)} = \dfrac{3x-20}{4(5-x)^{3/2}}$

$f''(x) < 0$ for $x < 5$, so f is CD on $(-\infty, 5)$. No IP

H.



20. $y = f(x) = 2\sqrt{x} - x$ **A.** $D = [0, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Rightarrow$

$2\sqrt{x} = x \quad \Rightarrow \quad 4x = x^2 \quad \Rightarrow \quad 4x - x^2 = 0 \quad \Rightarrow \quad x(4-x) = 0 \quad \Rightarrow \quad x = 0, 4$ **C.** No symmetry

D. No asymptote **E.** $f'(x) = \dfrac{1}{\sqrt{x}} - 1 = \dfrac{1}{\sqrt{x}}\left(1 - \sqrt{x}\right)$. This is positive for $x < 1$ and

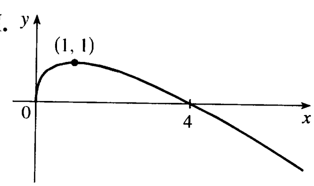
negative for $x > 1$, so f is increasing on $(0, 1)$ and decreasing on $(1, \infty)$.

F. Local maximum value $f(1) = 1$, no local minimum.

G. $f''(x) = \left(x^{-1/2} - 1\right)' = -\frac{1}{2}x^{-3/2} = \dfrac{-1}{2x^{3/2}} < 0$ for $x > 0$,

so f is CD on $(0, \infty)$. No IP

H.



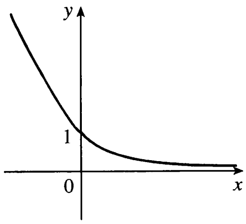
21. $y = f(x) = \sqrt{x^2+1} - x$ **A.** $D = \mathbb{R}$ **B.** No x-intercept, y-intercept $= 1$ **C.** No symmetry

D. $\lim\limits_{x \to -\infty} \left(\sqrt{x^2+1} - x\right) = \infty$ and

$\lim\limits_{x \to \infty} \left(\sqrt{x^2+1} - x\right) = \lim\limits_{x \to \infty} \left(\sqrt{x^2+1} - x\right) \dfrac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x} = \lim\limits_{x \to \infty} \dfrac{1}{\sqrt{x^2+1} + x} = 0$, so $y = 0$ is a HA. **E.** $f'(x) = \dfrac{x}{\sqrt{x^2+1}} - 1 = \dfrac{x - \sqrt{x^2+1}}{\sqrt{x^2+1}} \Rightarrow$ $f'(x) < 0$, so f is decreasing on $\mathbb{R}$. **F.** No extreme values

G. $f''(x) = \dfrac{1}{(x^2+1)^{3/2}} > 0$, so f is CU on $\mathbb{R}$. No IP

H.



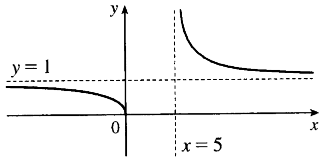
22. $y = f(x) = \sqrt{x/(x-5)}$ **A.** $D = \{x \mid x/(x-5) \geq 0\} = (-\infty, 0] \cup (5, \infty)$. **B.** Intercepts are 0.

C. No symmetry **D.** $\lim\limits_{x\to\pm\infty} \sqrt{\dfrac{x}{x-5}} = \lim\limits_{x\to\pm\infty} \sqrt{\dfrac{1}{1-5/x}} = 1$, so $y = 1$ is a HA. $\lim\limits_{x\to5^+} \sqrt{\dfrac{x}{x-5}} = \infty$,

so $x = 5$ is a VA. **E.** $f'(x) = \dfrac{1}{2}\left(\dfrac{x}{x-5}\right)^{-1/2} \dfrac{(-5)}{(x-5)^2} = -\frac{5}{2}\left[x(x-5)^3\right]^{-1/2} < 0$, so f is decreasing on $(-\infty, 0)$ and $(5, \infty)$. **F.** No extreme values

G. $f''(x) = \frac{5}{4}\left[x(x-5)^3\right]^{-3/2} (x-5)^2(4x-5) > 0$ for $x > 5$, and $f''(x) < 0$ for $x < 0$, so f is CU on $(5, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



23. $y = f(x) = x/\sqrt{x^2+1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 0$

C. $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D. $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2+1}/\sqrt{x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^2}}$$
$$= \frac{1}{\sqrt{1+0}} = 1$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}\Big/\left(-\sqrt{x^2}\right)}$$
$$= \lim_{x\to-\infty} \frac{1}{-\sqrt{1+1/x^2}} = \frac{1}{-\sqrt{1+0}} = -1$$

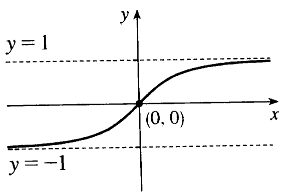
so $y = \pm 1$ are HA. No VA.

E. $$f'(x) = \frac{\sqrt{x^2+1} - x \cdot \dfrac{2x}{2\sqrt{x^2+1}}}{[(x^2+1)^{1/2}]^2} = \frac{x^2+1-x^2}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)^{3/2}}$$ > 0 for all x, so f is increasing on $\mathbb{R}$.

F. No extreme values

G. $f''(x) = -\frac{3}{2}(x^2+1)^{-5/2} \cdot 2x = \dfrac{-3x}{(x^2+1)^{5/2}}$, so $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$. Thus, f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $(0, 0)$

H.

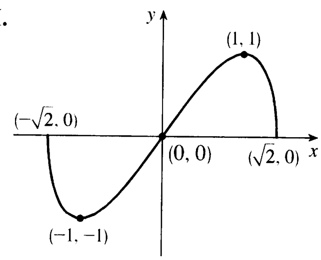


24. $y = f(x) = x\sqrt{2 - x^2}$ **A.** $D = \left[-\sqrt{2}, \sqrt{2}\right]$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Rightarrow$ $x = 0, \pm\sqrt{2}$. **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote

E. $f'(x) = x \cdot \dfrac{-x}{\sqrt{2 - x^2}} + \sqrt{2 - x^2} = \dfrac{-x^2 + 2 - x^2}{\sqrt{2 - x^2}} = \dfrac{2(1 + x)(1 - x)}{\sqrt{2 - x^2}}$. $f'(x)$ is negative for $-\sqrt{2} < x < -1$ and $1 < x < \sqrt{2}$, and positive for $-1 < x < 1$, so f is decreasing on $\left(-\sqrt{2}, -1\right)$ and $\left(1, \sqrt{2}\right)$ and increasing on $(-1, 1)$. **F.** Local minimum value $f(-1) = -1$, local maximum value $f(1) = 1$.

$$\textbf{G. } f''(x) = \frac{\sqrt{2 - x^2}(-4x) - (2 - 2x^2)\dfrac{-x}{\sqrt{2 - x^2}}}{\left[(2 - x^2)^{1/2}\right]^2} = \frac{(2 - x^2)(-4x) + (2 - 2x^2)x}{(2 - x^2)^{3/2}} = \frac{2x^3 - 6x}{(2 - x^2)^{3/2}} = \frac{2x(x^2 - 3)}{(2 - x^2)^{3/2}}$$

H.



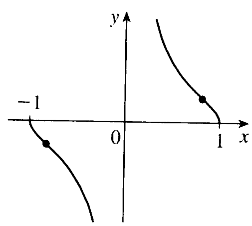
Since $x^2 - 3 < 0$ for x in $\left[-\sqrt{2}, \sqrt{2}\right]$, $f''(x) > 0$ for $-\sqrt{2} < x < 0$ and $f''(x) < 0$ for $0 < x < \sqrt{2}$. Thus, f is CU on $\left(-\sqrt{2}, 0\right)$ and CD on $\left(0, \sqrt{2}\right)$. The only IP is $(0, 0)$.

25. $y = f(x) = \sqrt{1 - x^2}/x$ **A.** $D = \left\{x \;\middle|\; |x| \le 1, x \ne 0\right\} = [-1, 0) \cup (0, 1]$ **B.** x-intercepts ± 1, no y-intercept

C. $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\displaystyle\lim_{x \to 0^+} \frac{\sqrt{1 - x^2}}{x} = \infty$, $\displaystyle\lim_{x \to 0^-} \frac{\sqrt{1 - x^2}}{x} = -\infty$, so $x = 0$ is a VA. **E.** $f'(x) = \dfrac{\left(-x^2/\sqrt{1 - x^2}\right) - \sqrt{1 - x^2}}{x^2} = -\dfrac{1}{x^2\sqrt{1 - x^2}} < 0$, so f is decreasing on $(-1, 0)$ and $(0, 1)$. **F.** No extreme values

G. $f''(x) = \dfrac{2 - 3x^2}{x^3(1 - x^2)^{3/2}} > 0 \;\Leftrightarrow\; -1 < x < -\sqrt{\frac{2}{3}}$ or $0 < x < \sqrt{\frac{2}{3}}$, so f is CU on $\left(-1, -\sqrt{\frac{2}{3}}\right)$ and $\left(0, \sqrt{\frac{2}{3}}\right)$ and CD on $\left(-\sqrt{\frac{2}{3}}, 0\right)$ and $\left(\sqrt{\frac{2}{3}}, 1\right)$. IP at $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{2}}\right)$

H.



26. $y = f(x) = x/\sqrt{x^2 - 1}$ **A.** $D = (-\infty, -1) \cup (1, \infty)$ **B.** No intercepts **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** $\displaystyle\lim_{x \to \infty} \frac{x}{\sqrt{x^2 - 1}} = 1$ and $\displaystyle\lim_{x \to -\infty} \frac{x}{\sqrt{x^2 - 1}} = -1$, so $y = \pm 1$ are HA.

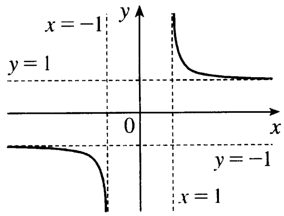
$\displaystyle\lim_{x \to 1^+} f(x) = +\infty$ and $\displaystyle\lim_{x \to -1^-} f(x) = -\infty$, so $x = \pm 1$ are VA.

E. $f'(x) = \dfrac{\sqrt{x^2 - 1} - x \cdot \dfrac{x}{\sqrt{x^2 - 1}}}{\left[(x^2 - 1)^{1/2}\right]^2} = \dfrac{x^2 - 1 - x^2}{(x^2 - 1)^{3/2}} = \dfrac{-1}{(x^2 - 1)^{3/2}} < 0$, so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$. **F.** No extreme values

G. $f''(x) = (-1)\left(-\frac{3}{2}\right)\left(x^2 - 1\right)^{-5/2} \cdot 2x = \dfrac{3x}{(x^2-1)^{5/2}}$.

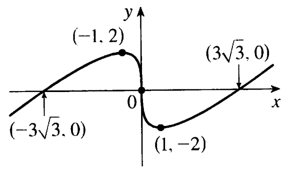
$f''(x) < 0$ on $(-\infty, -1)$ and $f''(x) > 0$ on $(1, \infty)$, so f is CD on $(-\infty, -1)$ and CU on $(1, \infty)$. No IP

H.



27. $y = f(x) = x - 3x^{1/3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 3x^{1/3} \Rightarrow x^3 = 27x \Rightarrow x^3 - 27x = 0 \Rightarrow x(x^2 - 27) = 0 \Rightarrow x = 0, \pm 3\sqrt{3}$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote **E.** $f'(x) = 1 - x^{-2/3} = 1 - \dfrac{1}{x^{2/3}} = \dfrac{x^{2/3} - 1}{x^{2/3}}$.
$f'(x) > 0$ when $|x| > 1$ and $f'(x) < 0$ when $0 < |x| < 1$, so f is increasing on $(-\infty, -1)$ and $(1, \infty)$, and decreasing on $(-1, 0)$ and $(0, 1)$ [hence decreasing on $(-1, 1)$ since f is continuous on $(-1, 1)$]. **F.** Local maximum value $f(-1) = 2$, local minimum value $f(1) = -2$ **G.** $f''(x) = \frac{2}{3}x^{-5/3} < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. IP at $(0, 0)$

H.



28. $y = f(x) = x^{5/3} - 5x^{2/3} = x^{2/3}(x - 5)$ **A.** $D = \mathbb{R}$ **B.** x-intercepts $0, 5$; y-intercept 0 **C.** No symmetry

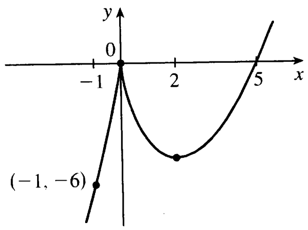
D. $\lim\limits_{x \to \pm\infty} x^{2/3}(x - 5) = \pm\infty$, so there is no asymptote

E. $f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x - 2) > 0 \Leftrightarrow x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$, $(2, \infty)$ and decreasing on $(0, 2)$.

F. Local maximum value $f(0) = 0$, local minimum value $f(2) = -3\sqrt[3]{4}$

G. $f''(x) = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x + 1) > 0 \Leftrightarrow x > -1$, so f is CU on $(-1, 0)$ and $(0, \infty)$, CD on $(-\infty, -1)$. IP at $(-1, -6)$

H.



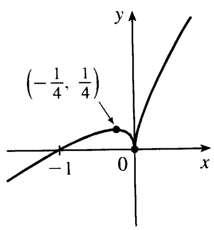
29. $y = f(x) = x + \sqrt{|x|}$ **A.** $D = \mathbb{R}$ **B.** x-intercepts $0, -1$; y-intercept 0 **C.** No symmetry

D. $\lim\limits_{x \to \infty}\left(x + \sqrt{|x|}\right) = \infty$, $\lim\limits_{x \to -\infty}\left(x + \sqrt{|x|}\right) = -\infty$. No asymptote **E.** For $x > 0$, $f(x) = x + \sqrt{x} \Rightarrow f'(x) = 1 + \dfrac{1}{2\sqrt{x}} > 0$, so f increases on $(0, \infty)$. For $x < 0$, $f(x) = x + \sqrt{-x} \Rightarrow f'(x) = 1 - \dfrac{1}{2\sqrt{-x}} > 0 \Leftrightarrow 2\sqrt{-x} > 1 \Leftrightarrow -x > \frac{1}{4} \Leftrightarrow x < -\frac{1}{4}$, so f increases on $\left(-\infty, -\frac{1}{4}\right)$ and decreases on $\left(-\frac{1}{4}, 0\right)$.

F. Local maximum value $f\left(-\frac{1}{4}\right) = \frac{1}{4}$, local minimum value $f(0) = 0$

G. For $x > 0$, $f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(x) < 0$, so f is CD on $(0, \infty)$. For $x < 0$, $f''(x) = -\frac{1}{4}(-x)^{-3/2} \Rightarrow f''(x) < 0$, so f is CD on $(-\infty, 0)$. No IP

H.



30. $y = f(x) = \sqrt[3]{(x^2-1)^2} = (x^2-1)^{2/3}$ **A.** $D = \mathbb{R}$ **B.** x-intercepts ± 1, y-intercept 1 **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim\limits_{x\to\pm\infty}(x^2-1)^{2/3} = \infty$, no asymptote

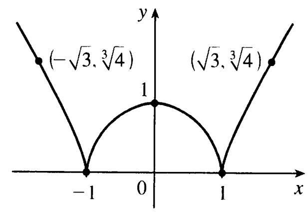
E. $f'(x) = \frac{4}{3}x(x^2-1)^{-1/3} \Rightarrow f'(x) > 0 \Leftrightarrow x > 1$ or $-1 < x < 0$, $f'(x) < 0 \Leftrightarrow x < -1$ or $0 < x < 1$. So f is increasing on $(-1,0)$, $(1,\infty)$ and decreasing on $(-\infty,-1)$, $(0,1)$. **F.** Local minimum values $f(-1) = f(1) = 0$, local maximum value $f(0) = 1$

G. $f''(x) = \frac{4}{3}\left(x^2-1\right)^{-1/3} + \frac{4}{3}x\left(-\frac{1}{3}\right)\left(x^2-1\right)^{-4/3}(2x)$

$= \frac{4}{9}\left(x^2-3\right)\left(x^2-1\right)^{-4/3} > 0 \Leftrightarrow |x| > \sqrt{3}$

so f is CU on $\left(-\infty, -\sqrt{3}\right)$, $\left(\sqrt{3},\infty\right)$ and CD on $\left(-\sqrt{3},-1\right)$, $\left(-1,1\right)$, $\left(1,\sqrt{3}\right)$. IPs at $\left(\pm\sqrt{3}, \sqrt[3]{4}\right)$

H.



31. $y = f(x) = 3\sin x - \sin^3 x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow$ $\sin x\,(3 - \sin^2 x) = 0 \Rightarrow \sin x = 0$ [since $\sin^2 x \le 1 < 3$] $\Rightarrow x = n\pi$, n an integer.

C. $f(-x) = -f(x)$, so f is odd; the graph (shown for $-2\pi \le x \le 2\pi$) is symmetric about the origin and periodic with period 2π. **D.** No asymptote **E.** $f'(x) = 3\cos x - 3\sin^2 x\cos x = 3\cos x\,(1-\sin^2 x) = 3\cos^3 x$.

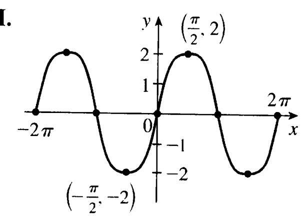
$f'(x) > 0 \Leftrightarrow \cos x > 0 \Leftrightarrow x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$ for each integer n, and $f'(x) < 0 \Leftrightarrow$ $\cos x < 0 \Leftrightarrow x \in \left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right)$ for each integer n. Thus, f is increasing on $\left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$ for each integer n, and f is decreasing on $\left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right)$ for each integer n.

F. f has local maximum values $f(2n\pi + \frac{\pi}{2}) = 2$ and local minimum values $f(2n\pi + \frac{3\pi}{2}) = -2$.

G. $f''(x) = -9\sin x\cos^2 x = -9\sin x\,(1-\sin^2 x) = -9\sin x\,(1-\sin x)(1+\sin x)$. $f''(x) < 0 \Leftrightarrow$ $\sin x > 0$ and $\sin x \ne \pm 1 \Leftrightarrow x \in \left(2n\pi, 2n\pi + \frac{\pi}{2}\right) \cup \left(2n\pi + \frac{\pi}{2}, 2n\pi + \pi\right)$ for some integer n.

$f''(x) > 0 \Leftrightarrow \sin x < 0$ and $\sin x \ne \pm 1 \Leftrightarrow x \in \left((2n-1)\pi, (2n-1)\pi + \frac{\pi}{2}\right) \cup \left((2n-1)\pi + \frac{\pi}{2}, 2n\pi\right)$ for some integer n. Thus, f is CD on the intervals $\left(2n\pi, \left(2n+\frac{1}{2}\right)\pi\right)$ and $\left(\left(2n+\frac{1}{2}\right)\pi, (2n+1)\,\pi\right)$ [hence CD on the intervals $(2n\pi, (2n+1)\,\pi)$] for each integer n, and f is CU on the intervals $\left((2n-1)\pi, \left(2n-\frac{1}{2}\right)\pi\right)$ and $\left(\left(2n-\frac{1}{2}\right)\pi, 2n\pi\right)$ [hence CU on the intervals $((2n-1)\pi, 2n\pi)$] for each integer n. f has inflection points at $(n\pi, 0)$ for each integer n.

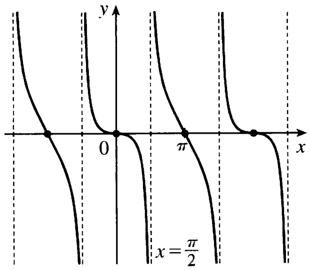
H.



32. $y = f(x) = \sin x - \tan x$ **A.** $D = \left\{x \mid x \ne (2n+1)\frac{\pi}{2}\right\}$ **B.** $y = 0 \Leftrightarrow \sin x = \tan x = \dfrac{\sin x}{\cos x} \Leftrightarrow$ $\sin x = 0$ or $\cos x = 1 \Leftrightarrow x = n\pi$ (x-intercepts), y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so the curve is symmetric about $(0,0)$. Also periodic with period 2π **D.** $\lim\limits_{x\to(\pi/2)^-}(\sin x - \tan x) = -\infty$ and $\lim\limits_{x\to(\pi/2)^+}(\sin x - \tan x) = \infty$, so $x = n\pi + \frac{\pi}{2}$ are VA.

E. $f'(x) = \cos x - \sec^2 x \le 0$, so f decreases on each interval in its domain, that is, on $\left((2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right)$. **F.** No extreme values **G.** $f''(x) = -\sin x - 2\sec^2 x \tan x = -\sin x\left(1 + 2\sec^3 x\right)$. Note that $1 + 2\sec^3 x \neq 0$ since $\sec^3 x \neq -\frac{1}{2}$. $f''(x) > 0$ for $-\frac{\pi}{2} < x < 0$ and $\frac{3\pi}{2} < x < 2\pi$, so f is CU on $\left(\left(n - \frac{1}{2}\right)\pi, n\pi\right)$ and CD on $\left(n\pi, \left(n + \frac{1}{2}\right)\pi\right)$. f has IPs at $(n\pi, 0)$. Note also that $f'(0) = 0$, but $f'(\pi) = -2$.

H.

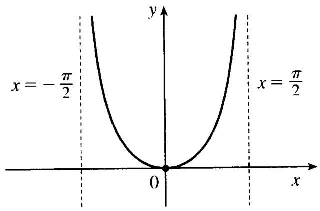


33. $y = f(x) = x\tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **B.** Intercepts are 0 **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim\limits_{x\to(\pi/2)^-} x\tan x = \infty$ and $\lim\limits_{x\to-(\pi/2)^+} x\tan x = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA. **E.** $f'(x) = \tan x + x\sec^2 x > 0 \iff 0 < x < \frac{\pi}{2}$, so f increases on $\left(0, \frac{\pi}{2}\right)$ and decreases on $\left(-\frac{\pi}{2}, 0\right)$.

F. Absolute and local minimum value $f(0) = 0$.

G. $y'' = 2\sec^2 x + 2x\tan x\sec^2 x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so f is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. No IP

H.



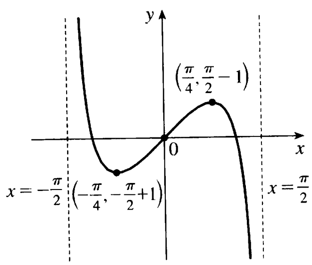
34. $y = f(x) = 2x - \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(0) = 0$ $\iff 2x = \tan x \iff x = 0$ or $x \approx \pm 1.17$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** $\lim\limits_{x\to(-\pi/2)^+}(2x - \tan x) = \infty$ and $\lim\limits_{x\to(\pi/2)^-}(2x - \tan x) = -\infty$, so $x = \pm\frac{\pi}{2}$ are VA. No HA.

E. $f'(x) = 2 - \sec^2 x < 0 \iff |\sec x| > \sqrt{2}$ and $f'(x) > 0 \iff |\sec x| < \sqrt{2}$, so f is decreasing on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$, increasing on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, and decreasing again on $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ **F.** Local maximum value $f\left(\frac{\pi}{4}\right) = \frac{\pi}{2} - 1$, local minimum value $f\left(-\frac{\pi}{4}\right) = -\frac{\pi}{2} + 1$

G. $f''(x) = -2\sec x \cdot \sec x\tan x = -2\tan x\sec^2 x$
$= -2\tan x(\tan^2 x + 1)$

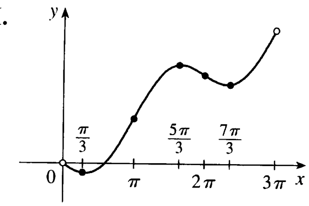
so $f''(x) > 0 \iff \tan x < 0 \iff -\frac{\pi}{2} < x < 0$, and $f''(x) < 0 \iff \tan x > 0 \iff 0 < x < \frac{\pi}{2}$. Thus, f is CU on $\left(-\frac{\pi}{2}, 0\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$. f has an IP at $(0, 0)$.

H.



35. $y = f(x) = \frac{1}{2}x - \sin x$, $0 < x < 3\pi$ **A.** $D = (0, 3\pi)$ **B.** No y-intercept. The x-intercept, approximately 1.9, can be found using Newton's Method. **C.** No symmetry **D.** No asymptote **E.** $f'(x) = \frac{1}{2} - \cos x > 0 \iff$ $\cos x < \frac{1}{2} \iff \frac{\pi}{3} < x < \frac{5\pi}{3}$ or $\frac{7\pi}{3} < x < 3\pi$, so f is increasing on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and $\left(\frac{7\pi}{3}, 3\pi\right)$ and decreasing on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{7\pi}{3}\right)$. **F.** Local minimum value $f\left(\frac{\pi}{3}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$, local maximum value $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{6} + \frac{\sqrt{3}}{2}$, local minimum value $f\left(\frac{7\pi}{3}\right) = \frac{7\pi}{6} - \frac{\sqrt{3}}{2}$ **G.** $f''(x) = \sin x > 0 \;\Leftrightarrow\; 0 < x < \pi$ or $2\pi < x < 3\pi$, so f is CU on $(0, \pi)$ and $(2\pi, 3\pi)$ and CD on $(\pi, 2\pi)$. IPs at $\left(\pi, \frac{\pi}{2}\right)$ and $(2\pi, \pi)$.

H.



36. $y = f(x) = \cos^2 x - 2\sin x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$ **C.** No symmetry, but f has period 2π. **D.** No asymptote **E.** $y' = 2\cos x\,(-\sin x) - 2\cos x = -2\cos x\,(\sin x + 1)$. $y' = 0 \;\Leftrightarrow\; \cos x = 0$ or $\sin x = -1 \;\Leftrightarrow\; x = (2n+1)\frac{\pi}{2}$. $y' > 0$ when $\cos x < 0$ since $\sin x + 1 \geq 0$ for all x. So $y' > 0$ and f is increasing on $\left((4n+1)\frac{\pi}{2}, (4n+3)\frac{\pi}{2}\right)$; $y' < 0$ and f is decreasing on $\left((4n-1)\frac{\pi}{2}, (4n+1)\frac{\pi}{2}\right)$. **F.** Local maximum values $f\left((4n+3)\frac{\pi}{2}\right) = 2$, local minimum values $f\left((4n+1)\frac{\pi}{2}\right) = -2$

G. $y' = -2\cos x\,(\sin x + 1) = -\sin 2x - 2\cos x \;\Rightarrow$

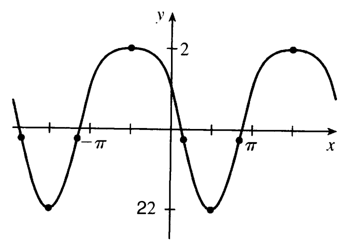
$$\begin{aligned} y'' &= -2\cos 2x + 2\sin x = -2\left(1 - 2\sin^2 x\right) + 2\sin x \\ &= 4\sin^2 x + 2\sin x - 2 = 2(2\sin x - 1)(\sin x + 1) \end{aligned}$$

$y'' = 0 \;\Leftrightarrow\; \sin x = \frac{1}{2}$ or $-1 \;\Rightarrow\; x = \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$, or $\frac{3\pi}{2} + 2n\pi$. $y'' > 0$ and f is CU on $\left(\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi\right)$;

$y'' \leq 0$ and f is CD on $\left(\frac{5\pi}{6} + 2n\pi, \frac{\pi}{6} + 2(n+1)\pi\right)$.

IPs at $\left(\frac{\pi}{6} + 2n\pi, -\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6} + 2n\pi, -\frac{1}{4}\right)$.

H.



37. $y = f(x) = \sin 2x - 2\sin x$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 0$. $y = 0 \;\Leftrightarrow$ $2\sin x = \sin 2x = 2\sin x\cos x \;\Leftrightarrow\; \sin x = 0$ or $\cos x = 1 \;\Leftrightarrow\; x = n\pi$ (x-intercepts)

C. $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$.

Note: f is periodic with period 2π, so we determine E–G for $-\pi \leq x \leq \pi$. **D.** No asymptotes

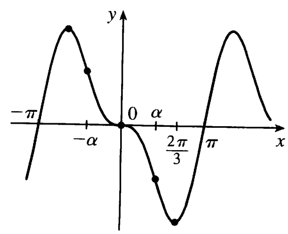
E. $f'(x) = 2\cos 2x - 2\cos x = 2\left(2\cos^2 x - 1 - \cos x\right) = 2(2\cos x + 1)(\cos x - 1) > 0 \;\Leftrightarrow\; \cos x < -\frac{1}{2}$ $\Leftrightarrow\; -\pi < x < -\frac{2\pi}{3}$ or $\frac{2\pi}{3} < x < \pi$, so f is increasing on $\left(-\pi, -\frac{2\pi}{3}\right)$, $\left(\frac{2\pi}{3}, \pi\right)$ and decreasing on $\left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)$.

F. Local maximum value $f\left(-\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{2}$, local minimum value $f\left(\frac{2\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$

G. $f''(x) = -4\sin 2x + 2\sin x = 2\sin x\,(1 - 4\cos x) = 0$ when $x = 0, \pm\pi$ or $\cos x = \frac{1}{4}$. If $\alpha = \cos^{-1}\frac{1}{4}$, then f is CU on $(-\alpha, 0)$ and (α, π) and CD on $(-\pi, -\alpha)$ and $(0, \alpha)$.

IPs at $(0, 0)$, $(\pm\pi, 0)$, $\left(\alpha, -\frac{3\sqrt{15}}{8}\right)$, $\left(-\alpha, \frac{3\sqrt{15}}{8}\right)$.

H.

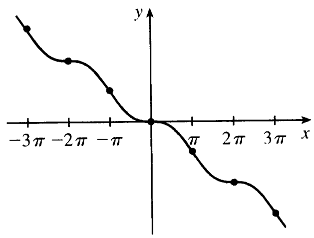


38. $f(x) = \sin x - x$ **A.** $D = \mathbb{R}$ **B.** x-intercept $= 0 =$ y-intercept

C. $f(-x) = \sin(-x) - (-x) = -(\sin x - x) = -f(x)$, so f is odd.

D. No asymptote **E.** $f'(x) = \cos x - 1 \leq 0$ for all x, so f is decreasing on $(-\infty, \infty)$. **F.** No extreme values **G.** $f''(x) = -\sin x \Rightarrow$ $f''(x) > 0 \Leftrightarrow \sin x < 0 \Leftrightarrow (2n-1)\pi < x < 2n\pi$, so f is CU on $((2n-1)\pi, 2n\pi)$ and CD on $(2n\pi, (2n+1)\pi)$, n an integer. Points of inflection occur when $x = n\pi$.

H.



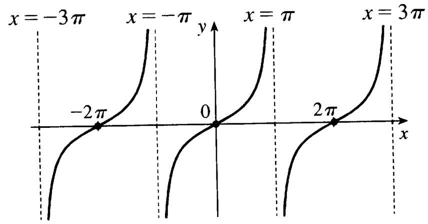
39. $y = f(x) = \dfrac{\sin x}{1 + \cos x}$ $\left[\overset{\text{when } \cos x \neq 1}{=} \dfrac{\sin x}{1+\cos x} \cdot \dfrac{1-\cos x}{1-\cos x} = \dfrac{\sin x\,(1-\cos x)}{\sin^2 x} = \dfrac{1-\cos x}{\sin x} = \csc x - \cot x \right]$

A. The domain of f is the set of all real numbers except odd integer multiples of π. **B.** y-intercept: $f(0) = 0$; x-intercepts: $x = n\pi$, n an even integer. **C.** $f(-x) = -f(x)$, so f is an odd function; the graph is symmetric about the origin and has period 2π. **D.** When n is an odd integer, $\lim\limits_{x \to (n\pi)^-} f(x) = \infty$ and $\lim\limits_{x \to (n\pi)^+} f(x) = -\infty$, so $x = n\pi$ is a VA for each odd integer n. No HA.

E. $f'(x) = \dfrac{(1+\cos x)\cdot \cos x - \sin x(-\sin x)}{(1+\cos x)^2} = \dfrac{1+\cos x}{(1+\cos x)^2} = \dfrac{1}{1+\cos x}$. $f'(x) > 0$ for all x except odd multiples of π, so f is increasing on $((2k-1)\pi, (2k+1)\pi)$ for each integer k. **F.** No extreme values

G. $f''(x) = \dfrac{\sin x}{(1+\cos x)^2} > 0 \Rightarrow \sin x > 0 \Rightarrow$ $x \in (2k\pi, (2k+1)\pi)$ and $f''(x) < 0$ on $((2k-1)\pi, 2k\pi)$ for each integer k. f is CU on $(2k\pi, (2k+1)\pi)$ and CD on $((2k-1)\pi, 2k\pi)$ for each integer k. f has IPs at $(2k\pi, 0)$ for each integer k.

H.



40. $y = f(x) = \cos x/(2 + \sin x)$ **A.** $D = \mathbb{R}$ *Note:* f is periodic with period 2π, so we determine B–G on $[0, 2\pi]$.

B. x-intercepts $\frac{\pi}{2}, \frac{3\pi}{2}$, y-intercept $= f(0) = \frac{1}{2}$ **C.** No symmetry other than periodicity **D.** No asymptote

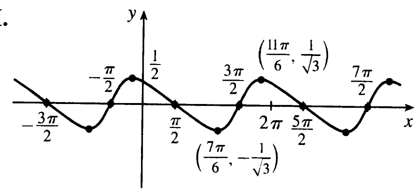
E. $f'(x) = \dfrac{(2+\sin x)(-\sin x) - \cos x\,(\cos x)}{(2+\sin x)^2} = -\dfrac{2\sin x + 1}{(2+\sin x)^2}$. $f'(x) > 0 \Leftrightarrow 2\sin x + 1 < 0 \Leftrightarrow$ $\sin x < -\frac{1}{2} \Leftrightarrow \frac{7\pi}{6} < x < \frac{11\pi}{6}$, so f is increasing on $\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$ and decreasing on $\left(0, \frac{7\pi}{6}\right)$, $\left(\frac{11\pi}{6}, 2\pi\right)$.

F. Local minimum value $f\left(\frac{7\pi}{6}\right) = -\frac{1}{\sqrt{3}}$, local maximum value $f\left(\frac{11\pi}{6}\right) = \frac{1}{\sqrt{3}}$

G. $f''(x) = -\dfrac{(2+\sin x)^2(2\cos x) - (2\sin x + 1)2(2+\sin x)\cos x}{(2+\sin x)^4} = -\dfrac{2\cos x\,(1-\sin x)}{(2+\sin x)^3} > 0 \quad\Leftrightarrow$

$\cos x < 0 \quad\Leftrightarrow\quad \frac{\pi}{2} < x < \frac{3\pi}{2}$, so f is CU on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$. IP at $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{2}, 0\right)$

H.



41. $y = 1/(1+e^{-x})$ **A.** $D = \mathbb{R}$ **B.** No x-intercept; y-intercept $= f(0) = \frac{1}{2}$. **C.** No symmetry

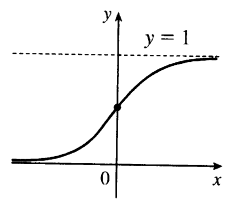
D. $\lim\limits_{x\to\infty} 1/(1+e^{-x}) = \frac{1}{1+0} = 1$ and $\lim\limits_{x\to-\infty} 1/(1+e^{-x}) = 0$ (since $\lim\limits_{x\to-\infty} e^{-x} = \infty$), so f has horizontal asymptotes $y = 0$ and $y = 1$. **E.** $f'(x) = -\left(1+e^{-x}\right)^{-2}\left(-e^{-x}\right) = e^{-x}/\left(1+e^{-x}\right)^2$. This is positive for all x, so f is increasing on $\mathbb{R}$. **F.** No extreme values

G. $f''(x) = \dfrac{\left(1+e^{-x}\right)^2\left(-e^{-x}\right) - e^{-x}(2)\left(1+e^{-x}\right)\left(-e^{-x}\right)}{(1+e^{-x})^4}$

$= \dfrac{e^{-x}\left(e^{-x}-1\right)}{(1+e^{-x})^3}$

The second factor in the numerator is negative for $x > 0$ and positive for $x < 0$, and the other factors are always positive, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. f has an inflection point at $\left(0, \frac{1}{2}\right)$.

H.

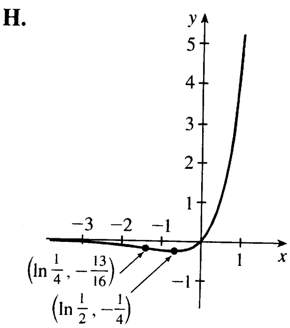


42. $y = f(x) = e^{2x} - e^x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Rightarrow\; e^{2x} = e^x \;\Rightarrow\; e^x = 1 \;\Rightarrow\; x = 0$.
C. No symmetry **D.** $\lim\limits_{x\to-\infty} e^{2x} - e^x = 0$, so $y = 0$ is a HA. No VA.

E. $f'(x) = 2e^{2x} - e^x = e^x(2e^x - 1)$, so $f'(x) > 0 \;\Leftrightarrow\; e^x > \frac{1}{2} \;\Leftrightarrow\; x > \ln\frac{1}{2} = -\ln 2$ and $f'(x) < 0 \;\Leftrightarrow\; e^x < \frac{1}{2} \;\Leftrightarrow\; x < \ln\frac{1}{2}$, so f is decreasing on $\left(-\infty, \ln\frac{1}{2}\right)$ and increasing on $\left(\ln\frac{1}{2}, \infty\right)$. **F.** Local minimum value $f\left(\ln\frac{1}{2}\right) = e^{2\ln(1/2)} - e^{\ln(1/2)} = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4}$

G. $f''(x) = 4e^{2x} - e^x = e^x(4e^x - 1)$, so $f''(x) > 0 \;\Leftrightarrow\; e^x > \frac{1}{4} \;\Leftrightarrow\; x > \ln\frac{1}{4}$ and $f''(x) < 0 \;\Leftrightarrow\; x < \ln\frac{1}{4}$.

H.



Thus, f is CD on $\left(-\infty, \ln\frac{1}{4}\right)$ and CU on $\left(\ln\frac{1}{4}, \infty\right)$. f has an IP at $\left(\ln\frac{1}{4}, \left(\frac{1}{4}\right)^2 - \frac{1}{4}\right) = \left(\ln\frac{1}{4}, -\frac{3}{16}\right)$.

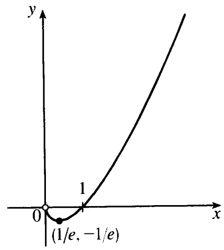
43. $y = f(x) = x \ln x$ **A.** $D = (0, \infty)$ **B.** x-intercept when $\ln x = 0 \Leftrightarrow x = 1$, no y-intercept

C. No symmetry **D.** $\lim\limits_{x \to \infty} x \ln x = \infty$,

$\lim\limits_{x \to 0^+} x \ln x = \lim\limits_{x \to 0^+} \dfrac{\ln x}{1/x} \overset{\text{H}}{=} \lim\limits_{x \to 0^+} \dfrac{1/x}{-1/x^2} = \lim\limits_{x \to 0^+} (-x) = 0$, no asymptote. **E.** $f'(x) = \ln x + 1 = 0$ when $\ln x = -1 \Leftrightarrow x = e^{-1}$.

$f'(x) > 0 \Leftrightarrow \ln x > -1 \Leftrightarrow x > e^{-1}$, so f is increasing on $(1/e, \infty)$ and decreasing on $(0, 1/e)$. **F.** $f(1/e) = -1/e$ is an absolute and local minimum value. **G.** $f''(x) = 1/x > 0$, so f is CU on $(0, \infty)$. No IP

H.



44. $y = f(x) = e^x/x$ **A.** $D = \{x \mid x \neq 0\}$ **B.** No intercept **C.** No symmetry **D.** $\lim\limits_{x \to \infty} \dfrac{e^x}{x} \overset{\text{H}}{=} \lim\limits_{x \to \infty} \dfrac{e^x}{1} = \infty$,

$\lim\limits_{x \to -\infty} \dfrac{e^x}{x} = 0$, so $y = 0$ is a HA. $\lim\limits_{x \to 0^+} \dfrac{e^x}{x} = \infty$, $\lim\limits_{x \to 0^-} \dfrac{e^x}{x} = -\infty$, so $x = 0$ is a VA.

E. $f'(x) = \dfrac{xe^x - e^x}{x^2} > 0 \Leftrightarrow (x-1)e^x > 0 \Leftrightarrow x > 1$,

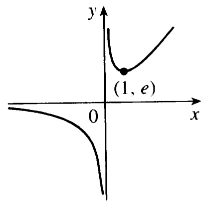
so f is increasing on $(1, \infty)$, and decreasing on $(-\infty, 0)$ and $(0, 1)$.

F. $f(1) = e$ is a local minimum value.

G. $f''(x) = \dfrac{x^2(xe^x) - 2x(xe^x - e^x)}{x^4} = \dfrac{e^x(x^2 - 2x + 2)}{x^3} > 0$

$\Leftrightarrow x > 0$ since $x^2 - 2x + 2 > 0$ for all x. So f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



45. $y = f(x) = xe^{-x}$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** No symmetry

D. $\lim\limits_{x \to \infty} xe^{-x} = \lim\limits_{x \to \infty} \dfrac{x}{e^x} \overset{\text{H}}{=} \lim\limits_{x \to \infty} \dfrac{1}{e^x} = 0$, so $y = 0$ is a HA.

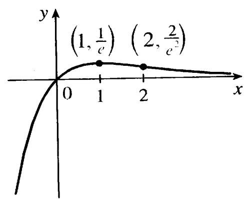
$\lim\limits_{x \to -\infty} xe^{-x} = -\infty$ **E.** $f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x) > 0 \Leftrightarrow$

$x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.

F. Absolute and local maximum value $f(1) = 1/e$.

G. $f''(x) = e^{-x}(x - 2) > 0 \Leftrightarrow x > 2$, so f is CU on $(2, \infty)$ and CD on $(-\infty, 2)$. IP at $(2, 2/e^2)$

H.



46. $y = f(x) = \ln(x^2 - 3x + 2) = \ln[(x-1)(x-2)]$

A. $D = \{x \text{ in } \mathbb{R}\colon x^2 - 3x + 2 > 0\} = (-\infty, 1) \cup (2, \infty)$.

B. y-intercept: $f(0) = \ln 2$; x-intercepts: $f(x) = 0 \Leftrightarrow x^2 - 3x + 2 = e^0 \Leftrightarrow x^2 - 3x + 1 = 0 \Leftrightarrow$

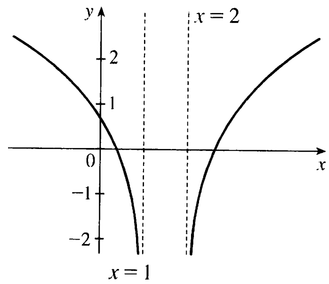
$x = \dfrac{3 \pm \sqrt{5}}{2} \Rightarrow x \approx 0.38, 2.62$ **C.** No symmetry **D.** $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 2^+} f(x) = -\infty$, so $x = 1$ and $x = 2$ are VAs. No HA. **E.** $f'(x) = \dfrac{2x - 3}{x^2 - 3x + 2} = \dfrac{2(x - 3/2)}{(x-1)(x-2)}$, so $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$

for $x > 2$. Thus, f is decreasing on $(-\infty, 1)$ and increasing on $(2, \infty)$. **F.** No extreme values

G. $$f''(x) = \frac{(x^2 - 3x + 2) \cdot 2 - (2x - 3)^2}{(x^2 - 3x + 2)^2} = \frac{2x^2 - 6x + 4 - 4x^2 + 12x - 9}{(x^2 - 3x + 2)^2} = \frac{-2x^2 + 6x - 5}{(x^2 - 3x + 2)^2}$$

The numerator is negative for all x and the denominator is positive, so $f''(x) < 0$ for all x in the domain of f. Thus, f is CD on $(-\infty, 1)$ and $(2, \infty)$. No IP

H.

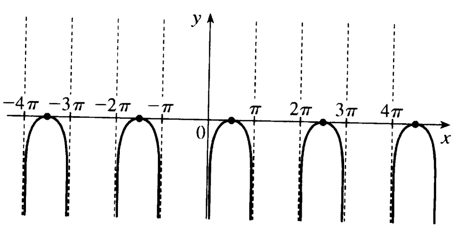


47. $y = f(x) = \ln(\sin x)$

A. $D = \{x \text{ in } \mathbb{R} \mid \sin x > 0\} = \bigcup_{n=-\infty}^{\infty} (2n\pi, (2n+1)\pi)$
$= \cdots \cup (-4\pi, -3\pi) \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \cdots$

B. No y-intercept; x-intercepts: $f(x) = 0 \Leftrightarrow \ln(\sin x) = 0 \Leftrightarrow \sin x = e^0 = 1 \Leftrightarrow x = 2n\pi + \frac{\pi}{2}$ for each integer n. **C.** f is periodic with period 2π. **D.** $\lim_{x \to (2n\pi)^+} f(x) = -\infty$ and $\lim_{x \to [(2n+1)\pi]^-} f(x) = -\infty$, so the lines $x = n\pi$ are VAs for all integers n. **E.** $f'(x) = \frac{\cos x}{\sin x} = \cot x$, so $f'(x) > 0$ when $2n\pi < x < 2n\pi + \frac{\pi}{2}$ for each integer n, and $f'(x) < 0$ when $2n\pi + \frac{\pi}{2} < x < (2n+1)\pi$. Thus, f is increasing on $\left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$ and decreasing on $\left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right)$ for each integer n. **F.** Local maximum values $f\left(2n\pi + \frac{\pi}{2}\right) = 0$, no local minimum. **G.** $f''(x) = -\csc^2 x < 0$, so f is CD on $(2n\pi, (2n+1)\pi)$ for each integer n. No IP

H.



48. $y = f(x) = x(\ln x)^2$ **A.** $D = (0, \infty)$ **B.** x-intercept $= 1$, no y-intercept **C.** No symmetry

D. $\lim_{x \to \infty} x(\ln x)^2 = \infty$, $\lim_{x \to 0^+} x(\ln x)^2 = \lim_{x \to 0^+} \frac{(\ln x)^2}{1/x} \overset{\text{H}}{=} \lim_{x \to 0^+} \frac{2(\ln x)(1/x)}{-1/x^2} = \lim_{x \to 0^+} \frac{2 \ln x}{-1/x} \overset{\text{H}}{=}$
$\lim_{x \to 0^+} \frac{2/x}{1/x^2} = \lim_{x \to 0^+} 2x = 0$, no asymptote **E.** $f'(x) = (\ln x)^2 + 2 \ln x = (\ln x)(\ln x + 2) = 0$ when $\ln x = 0$ $\Leftrightarrow x = 1$ and when $\ln x = -2 \Leftrightarrow x = e^{-2}$. $f'(x) > 0$ when $0 < x < e^{-2}$ and when $x > 1$, so f is increasing on $(0, e^{-2})$ and $(1, \infty)$ and decreasing on $(e^{-2}, 1)$.

F. Local maximum value $f(e^{-2}) = 4e^{-2}$,

local minimum value $f(1) = 0$

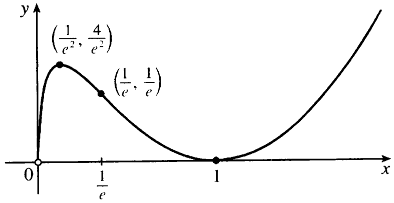
G. $f''(x) = 2(\ln x)(1/x) + 2/x = (2/x)(\ln x + 1) = 0$

when $\ln x = -1 \iff x = e^{-1}$. $f''(x) > 0 \iff$

$x > 1/e$, so f is CU on $(1/e, \infty)$, CD on $(0, 1/e)$.

IP at $(1/e, 1/e)$

H.



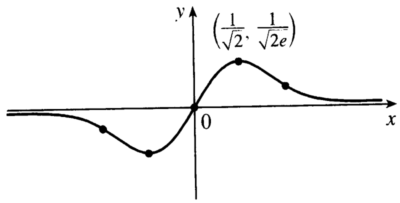
49. $y = f(x) = xe^{-x^2}$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\displaystyle\lim_{x\to\pm\infty} xe^{-x^2} = \lim_{x\to\pm\infty} \frac{x}{e^{x^2}} \overset{\text{H}}{=} \lim_{x\to\pm\infty} \frac{1}{2xe^{x^2}} = 0$, so $y = 0$ is a HA.

E. $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1 - 2x^2) > 0 \iff x^2 < \frac{1}{2} \iff |x| < \frac{1}{\sqrt{2}}$, so f is increasing on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and decreasing on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$. **F.** Local maximum value $f\left(\frac{1}{\sqrt{2}}\right) = 1/\sqrt{2e}$, local minimum value $f\left(-\frac{1}{\sqrt{2}}\right) = -1/\sqrt{2e}$ **G.** $f''(x) = -2xe^{-x^2}(1 - 2x^2) - 4xe^{-x^2} = 2xe^{-x^2}(2x^2 - 3) > 0$

$\iff x > \sqrt{\frac{3}{2}}$ or $-\sqrt{\frac{3}{2}} < x < 0$, so f is CU on $\left(\sqrt{\frac{3}{2}}, \infty\right)$ and $\left(-\sqrt{\frac{3}{2}}, 0\right)$ and CD on $\left(-\infty, -\sqrt{\frac{3}{2}}\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$.

IP are $(0, 0)$ and $\left(\pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{3}{2}}\, e^{-3/2}\right)$.

H.



50. $y = f(x) = e^x - 3e^{-x} - 4x$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= -2$; x-intercept ≈ 2.22 **C.** No symmetry

D. $\displaystyle\lim_{x\to\infty}(e^x - 3e^{-x} - 4x) = \lim_{x\to\infty} x\left(\frac{e^x}{x} - 3\frac{e^{-x}}{x} - 4\right) = \infty$, since $\displaystyle\lim_{x\to\infty}\frac{e^x}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{1} = \infty$.

Similarly, $\displaystyle\lim_{x\to-\infty}(e^x - 3e^{-x} - 4x) = -\infty$. No HA; no VA

E. $f'(x) = e^x + 3e^{-x} - 4 = e^{-x}(e^{2x} - 4e^x + 3) = e^{-x}(e^x - 3)(e^x - 1) > 0 \iff e^x > 3$ or $e^x < 1 \iff$

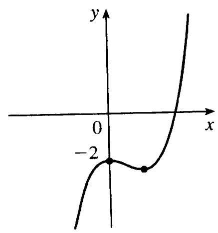
$x > \ln 3$ or $x < 0$. So f is increasing on $(-\infty, 0)$ and $(\ln 3, \infty)$ and decreasing on $(0, \ln 3)$. **F.** Local maximum value $f(0) = -2$, local minimum value $f(\ln 3) = 2 - 4\ln 3$

G. $f''(x) = e^x - 3e^{-x} = e^{-x}(e^{2x} - 3) > 0 \iff e^{2x} > 3 \iff$

$x > \frac{1}{2}\ln 3$, so f is CU on $\left(\frac{1}{2}\ln 3, \infty\right)$ and CD on $\left(-\infty, \frac{1}{2}\ln 3\right)$.

IP at $\left(\frac{1}{2}\ln 3, -2\ln 3\right)$.

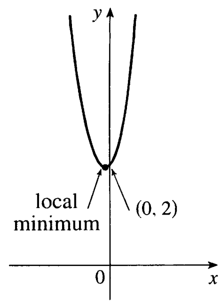
H.



51. $y = f(x) = e^{3x} + e^{-2x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 2$; no x-intercept **C.** No symmetry **D.** No asymptotes

E. $f'(x) = 3e^{3x} - 2e^{-2x}$, so $f'(x) > 0 \Leftrightarrow 3e^{3x} > 2e^{-2x}$ [multiply by e^{2x}] $\Leftrightarrow e^{5x} > \frac{2}{3} \Leftrightarrow 5x > \ln\frac{2}{3} \Leftrightarrow$ $x > \frac{1}{5}\ln\frac{2}{3} \approx -0.081$. Similarly, $f'(x) < 0 \Leftrightarrow x < \frac{1}{5}\ln\frac{2}{3}$. f is decreasing on $\left(-\infty, \frac{1}{5}\ln\frac{2}{3}\right)$ and increasing on $\left(\frac{1}{5}\ln\frac{2}{3}, \infty\right)$.

H.



F. Local minimum value $f\left(\frac{1}{5}\ln\frac{2}{3}\right) = \left(\frac{2}{3}\right)^{3/5} + \left(\frac{2}{3}\right)^{-2/5} \approx 1.96$; no local maximum.

G. $f''(x) = 9e^{3x} + 4e^{-2x}$, so $f''(x) > 0$ for all x, and f is CU on $(-\infty, \infty)$. No IP

52. $y = f(x) = \tan^{-1}\left(\dfrac{x-1}{x+1}\right)$ **A.** $D = \{x \mid x \neq -1\}$

B. x-intercept $= 1$, y-intercept $= f(0) = \tan^{-1}(-1) = -\frac{\pi}{4}$ **C.** No symmetry

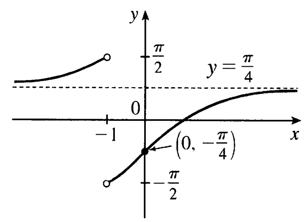
D. $\displaystyle\lim_{x\to\pm\infty} \tan^{-1}\left(\frac{x-1}{x+1}\right) = \lim_{x\to\pm\infty} \tan^{-1}\left(\frac{1-1/x}{1+1/x}\right) = \tan^{-1} 1 = \frac{\pi}{4}$, so $y = \frac{\pi}{4}$ is a HA. Also

$\displaystyle\lim_{x\to -1^+} \tan^{-1}\left(\frac{x-1}{x+1}\right) = -\frac{\pi}{2}$ and $\displaystyle\lim_{x\to -1^-} \tan^{-1}\left(\frac{x-1}{x+1}\right) = \frac{\pi}{2}$.

E.
$$f'(x) = \frac{1}{1 + [(x-1)/(x+1)]^2}\,\frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2 + (x-1)^2} = \frac{1}{x^2+1} > 0$$

so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values

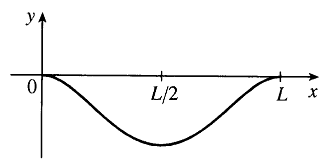
H.



G. $f''(x) = -2x/(x^2+1)^2 > 0 \Leftrightarrow x < 0$, so f is CU on $(-\infty, -1)$ and $(-1, 0)$, and CD on $(0, \infty)$. IP at $\left(0, -\frac{\pi}{4}\right)$

53.
$$y = -\frac{W}{24EI}x^4 + \frac{WL}{12EI}x^3 - \frac{WL^2}{24EI}x^2 = -\frac{W}{24EI}x^2(x^2 - 2Lx + L^2) = \frac{-W}{24EI}x^2(x-L)^2 = cx^2(x-L)^2$$

where $c = -\dfrac{W}{24EI}$ is a negative constant and $0 \le x \le L$. We sketch $f(x) = cx^2(x-L)^2$ for $c = -1$. $f(0) = f(L) = 0$.



$f'(x) = cx^2[2(x-L)] + (x-L)^2(2cx) = 2cx(x-L)[x + (x-L)] = 2cx(x-L)(2x-L)$. So for $0 < x < L$, $f'(x) > 0 \Leftrightarrow x(x-L)(2x-L) < 0$ (since $c < 0$) $\Leftrightarrow L/2 < x < L$ and $f'(x) < 0 \Leftrightarrow$ $0 < x < L/2$. So f is increasing on $(L/2, L)$ and decreasing on $(0, L/2)$, and there is a local and absolute minimum at $(L/2, f(L/2)) = \left(L/2, cL^4/16\right)$.

$f'(x) = 2c\,[x(x-L)(2x-L)] \quad \Rightarrow$

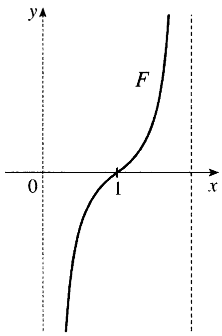
$f''(x) = 2c\,[1(x-L)(2x-L) + x(1)(2x-L) + x(x-L)(2)] = 2c\left(6x^2 - 6Lx + L^2\right) = 0 \quad \Leftrightarrow$

$x = \dfrac{6L \pm \sqrt{12L^2}}{12} = \frac{1}{2}L \pm \frac{\sqrt{3}}{6}L$, and these are the x-coordinates of the two inflection points.

54. $F(x) = -\dfrac{k}{x^2} + \dfrac{k}{(x-2)^2}$, where $k > 0$ and $0 < x < 2$. For $0 < x < 2$,

$x - 2 < 0$, so $F'(x) = \dfrac{2k}{x^3} - \dfrac{2k}{(x-2)^3} > 0$ and F is increasing.

$\lim\limits_{x \to 0^+} F(x) = -\infty$ and $\lim\limits_{x \to 2^-} F(x) = \infty$, so $x = 0$ and $x = 2$ are vertical asymptotes. Notice that when the middle particle is at $x = 1$, the net force acting on it is 0. When $x > 1$, the net force is positive, meaning that it acts to the right. And if the particle approaches $x = 2$, the force on it rapidly becomes very large. When $x < 1$, the net force is negative, so it acts to the left. If the particle approaches 0, the force becomes very large to the left.



55. $y = \dfrac{x^2+1}{x+1}$. Long division gives us:

$$
\begin{array}{r|l}
 & \quad x - 1 \\
\hline
x+1 & x^2 \qquad + 1 \\
 & x^2 + x \\
\hline
 & \quad -x + 1 \\
 & \quad -x - 1 \\
\hline
 & \qquad\quad 2
\end{array}
$$

Thus, $y = f(x) = \dfrac{x^2+1}{x+1} = x - 1 + \dfrac{2}{x+1}$ and $f(x) - (x-1) = \dfrac{2}{x+1} = \dfrac{\frac{2}{x}}{1 + \frac{1}{x}}$ [for $x \neq 0$] $\to 0$

as $x \to \pm\infty$. So the line $y = x - 1$ is a slant asymptote (SA).

56. $y = \dfrac{2x^3 + x^2 + x + 3}{x^2 + 2x}$. Long division gives us:

$$
\begin{array}{r|l}
 & \qquad\qquad 2x - 3 \\
\hline
x^2 + 2x & 2x^3 + \ x^2 + \ x + 3 \\
 & 2x^3 + 4x^2 \\
\hline
 & \quad -3x^2 + \ x \\
 & \quad -3x^2 - 6x \\
\hline
 & \qquad\qquad 7x + 3
\end{array}
$$

Thus, $y = f(x) = \dfrac{2x^3 + x^2 + x + 3}{x^2 + 2x} = 2x - 3 + \dfrac{7x+3}{x^2+2x}$ and $f(x) - (2x - 3) = \dfrac{7x+3}{x^2+2x} = \dfrac{\frac{7}{x} + \frac{3}{x^2}}{1 + \frac{2}{x}}$

[for $x \neq 0$] $\to 0$ as $x \to \pm\infty$. So the line $y = 2x - 3$ is a SA.

57. $y = \dfrac{4x^3 - 2x^2 + 5}{2x^2 + x - 3}$. Long division gives us:

$$\begin{array}{r|l} & 2x - 2 \\ \hline 2x^2 + x - 3 & 4x^3 - 2x^2 \qquad\quad + 5 \\ & 4x^3 + 2x^2 - 6x \\ \hline & \quad - 4x^2 + 6x + 5 \\ & \quad - 4x^2 - 2x + 6 \\ \hline & \qquad\qquad 8x - 1 \end{array}$$

Thus, $y = f(x) = \dfrac{4x^3 - 2x^2 + 5}{2x^2 + x - 3} = 2x - 2 + \dfrac{8x - 1}{2x^2 + x - 3}$ and

$f(x) - (2x - 2) = \dfrac{8x - 1}{2x^2 + x - 3} = \dfrac{\dfrac{8}{x} - \dfrac{1}{x^2}}{2 + \dfrac{1}{x} - \dfrac{3}{x^2}}$ [for $x \neq 0$] $\to 0$ as $x \to \pm\infty$. So the line $y = 2x - 2$ is a SA.

58. $y = \dfrac{5x^4 + x^2 + x}{x^3 - x^2 + 2}$. Long division gives us:

$$\begin{array}{r|l} & 5x + 5 \\ \hline x^3 - x^2 + 2 & 5x^4 \qquad\quad + x^2 + x \\ & 5x^4 - 5x^3 \qquad + 10x \\ \hline & \qquad 5x^3 + x^2 - 9x \\ & \qquad 5x^3 - 5x^2 \qquad + 10 \\ \hline & \qquad\qquad 6x^2 - 9x - 10 \end{array}$$

Thus, $y = f(x) = \dfrac{5x^4 + x^2 + x}{x^3 - x^2 + 2} = 5x + 5 + \dfrac{6x^2 - 9x - 10}{x^3 - x^2 + 2}$ and

$f(x) - (5x + 5) = \dfrac{6x^2 - 9x - 10}{x^3 - x^2 + 2} = \dfrac{\dfrac{6}{x} - \dfrac{9}{x^2} - \dfrac{10}{x^3}}{1 - \dfrac{1}{x} + \dfrac{2}{x^3}}$ [for $x \neq 0$] $\to 0$ as $x \to \pm\infty$. So the line $y = 5x + 5$ is a SA.

59. $y = f(x) = \dfrac{-2x^2 + 5x - 1}{2x - 1} = -x + 2 + \dfrac{1}{2x - 1}$ **A.** $D = \{x \in \mathbb{R} \mid x \neq \frac{1}{2}\} = (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$

B. y-intercept: $f(0) = 1$; x-intercepts: $f(x) = 0 \Rightarrow -2x^2 + 5x - 1 = 0 \Rightarrow x = \dfrac{-5 \pm \sqrt{17}}{-4} \Rightarrow$ $x \approx 0.22, 2.28$. **C.** No symmetry

D. $\lim\limits_{x \to (1/2)^-} f(x) = -\infty$ and $\lim\limits_{x \to (1/2)^+} f(x) = \infty$, so $x = \frac{1}{2}$ is a VA.

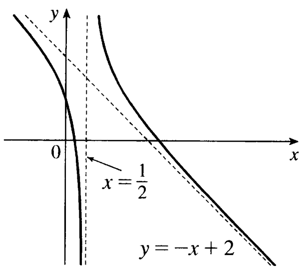
$\lim\limits_{x \to \pm\infty} [f(x) - (-x + 2)] = \lim\limits_{x \to \pm\infty} \dfrac{1}{2x - 1} = 0$, so the line $y = -x + 2$ is a SA.

E. $f'(x) = -1 - \dfrac{2}{(2x-1)^2} < 0$ for $x \neq \frac{1}{2}$, so f is decreasing on $\left(-\infty, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \infty\right)$. **F.** No extreme values

G. $f'(x) = -1 - 2(2x-1)^{-2} \Rightarrow$

$f''(x) = -2(-2)(2x-1)^{-3}(2) = \dfrac{8}{(2x-1)^3}$, so $f''(x) > 0$ when $x > \frac{1}{2}$ and $f''(x) < 0$ when $x < \frac{1}{2}$. Thus, f is CU on $\left(\frac{1}{2}, \infty\right)$ and CD on $\left(-\infty, \frac{1}{2}\right)$. No IP

H.



60. $y = f(x) = \dfrac{x^2+12}{x-2} = x + 2 + \dfrac{16}{x-2}$ **A.** $D = \{x \in \mathbb{R} \mid x \neq 2\} = (-\infty, 2) \cup (2, \infty)$

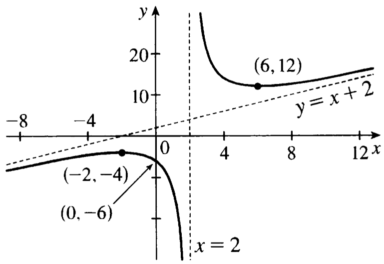
B. y-intercept: $f(0) = -6$; no x-intercepts. **C.** No symmetry **D.** $\lim\limits_{x \to 2^-} f(x) = -\infty$ and $\lim\limits_{x \to 2^+} f(x) = \infty$, so $x = 2$ is a VA. $\lim\limits_{x \to \pm\infty} [f(x) - (x+2)] = \lim\limits_{x \to \pm\infty} \dfrac{16}{x-2} = 0$, so the line $y = x + 2$ is a slant asymptote.

E. $f'(x) = 1 - \dfrac{16}{(x-2)^2} = \dfrac{x^2 - 4x - 12}{(x-2)^2} = \dfrac{(x-6)(x+2)}{(x-2)^2}$, so $f'(x) > 0$ when $x < -2$ or $x > 6$ and $f'(x) < 0$ when $-2 < x < 2$ or $2 < x < 6$. Thus, f is increasing on $(-\infty, -2)$ and $(6, \infty)$ and decreasing on $(-2, 2)$ and $(2, 6)$.

F. Local maximum value $f(-2) = -4$, local minimum value $f(6) = 12$

G. $f''(x) = 16(-2)(x-2)^{-3} = \dfrac{32}{(x-2)^3}$, so $f''(x) > 0$ for $x > 2$ and $f''(x) < 0$ for $x < 2$. f is CU on $(2, \infty)$ and CD on $(-\infty, 2)$. No IP

H.



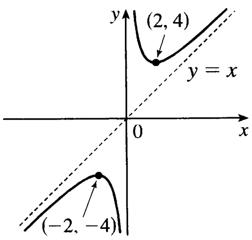
61. $y = f(x) = (x^2 + 4)/x = x + 4/x$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No intercept

C. $f(-x) = -f(x) \Rightarrow$ symmetry about the origin **D.** $\lim\limits_{x \to \infty} (x + 4/x) = \infty$ but $f(x) - x = 4/x \to 0$ as $x \to \pm\infty$, so $y = x$ is a slant asymptote. $\lim\limits_{x \to 0^+} (x + 4/x) = \infty$ and $\lim\limits_{x \to 0^-} (x + 4/x) = -\infty$, so $x = 0$ is a VA. **E.** $f'(x) = 1 - 4/x^2 > 0$ $\Leftrightarrow$ $x^2 > 4$ $\Leftrightarrow$ $x > 2$ or $x < -2$, so f is increasing on $(-\infty, -2)$ and $(2, \infty)$ and decreasing on $(-2, 0)$ and $(0, 2)$.

F. Local maximum value $f(-2) = -4$, local minimum value $f(2) = 4$

G. $f''(x) = 8/x^3 > 0$ $\Leftrightarrow$ $x > 0$ so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



62. $y = f(x) = e^x - x$ **A.** $D = \mathbb{R}$ **B.** No x-intercept; y-intercept $= 1$ **C.** No symmetry

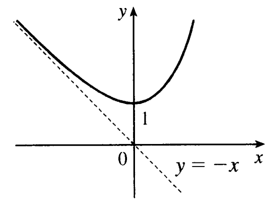
D. $\lim\limits_{x\to-\infty}(e^x - x) = \infty$, $\lim\limits_{x\to\infty}(e^x - x) = \lim\limits_{x\to\infty} x\left(\dfrac{e^x}{x} - 1\right) = \infty$ since $\lim\limits_{x\to\infty}\dfrac{e^x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{e^x}{1} = \infty$.

$y = -x$ is a slant asymptote since $(e^x - x) - (-x) = e^x \to 0$ as $x \to -\infty$. **E.** $f'(x) = e^x - 1 > 0 \Leftrightarrow e^x > 1 \Leftrightarrow x > 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

F. $f(0) = 1$ is a local and absolute minimum value.

G. $f''(x) = e^x > 0$ for all x, so f is CU on $\mathbb{R}$. No IP

H.



63. $y = f(x) = \dfrac{2x^3 + x^2 + 1}{x^2 + 1} = 2x + 1 + \dfrac{-2x}{x^2 + 1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$; x-intercept: $f(x) = 0$ $\Rightarrow$ $0 = 2x^3 + x^2 + 1 = (x + 1)(2x^2 - x + 1)$ $\Rightarrow$ $x = -1$ **C.** No symmetry **D.** No VA

$\lim\limits_{x\to\pm\infty}[f(x) - (2x + 1)] = \lim\limits_{x\to\pm\infty}\dfrac{-2x}{x^2 + 1} = \lim\limits_{x\to\pm\infty}\dfrac{-2/x}{1 + 1/x^2} = 0$, so the line $y = 2x + 1$ is a slant asymptote.

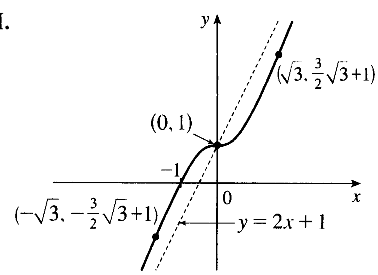
$$\textbf{E. } f'(x) = 2 + \frac{(x^2 + 1)(-2) - (-2x)(2x)}{(x^2 + 1)^2} = \frac{2(x^4 + 2x^2 + 1) - 2x^2 - 2 + 4x^2}{(x^2 + 1)^2} = \frac{2x^4 + 6x^2}{(x^2 + 1)^2} = \frac{2x^2(x^2 + 3)}{(x^2 + 1)^2}$$

so $f'(x) > 0$ if $x \neq 0$. Thus, f is increasing on $(-\infty, 0)$ and $(0, \infty)$. Since f is continuous at 0, f is increasing on $\mathbb{R}$. **F.** No extreme values

$$\textbf{G. } f''(x) = \frac{(x^2 + 1)^2 \cdot (8x^3 + 12x) - (2x^4 + 6x^2) \cdot 2(x^2 + 1)(2x)}{[(x^2 + 1)^2]^2} = \frac{4x(x^2 + 1)[(x^2 + 1)(2x^2 + 3) - 2x^4 - 6x^2]}{(x^2 + 1)^4} = \frac{4x(-x^2 + 3)}{(x^2 + 1)^3}$$

so $f''(x) > 0$ for $x < -\sqrt{3}$ and $0 < x < \sqrt{3}$, and $f''(x) < 0$ for $-\sqrt{3} < x < 0$ and $x > \sqrt{3}$. f is CU on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$, and CD on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$. There are three IPs: $(0, 1)$, $\left(-\sqrt{3}, -\frac{3}{2}\sqrt{3} + 1\right) \approx (-1.73, -1.60)$, and $\left(\sqrt{3}, \frac{3}{2}\sqrt{3} + 1\right) \approx (1.73, 3.60)$.

H.



64. $y = f(x) = \dfrac{(x + 1)^3}{(x - 1)^2} = \dfrac{x^3 + 3x^2 + 3x + 1}{x^2 - 2x + 1} = x + 5 + \dfrac{12x - 4}{(x - 1)^2}$

A. $D = \{x \in \mathbb{R} \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ **B.** y-intercept: $f(0) = 1$; x-intercept: $f(x) = 0$ $\Rightarrow$ $x = -1$ **C.** No symmetry **D.** $\lim\limits_{x\to1} f(x) = \infty$, so $x = 1$ is a VA.

$\lim\limits_{x\to\pm\infty}[f(x) - (x + 5)] = \lim\limits_{x\to\pm\infty}\dfrac{12x - 4}{x^2 - 2x + 1} = \lim\limits_{x\to\pm\infty}\dfrac{\dfrac{12}{x} - \dfrac{4}{x^2}}{1 - \dfrac{2}{x} + \dfrac{1}{x^2}} = 0$, so the line $y = x + 5$ is a SA.

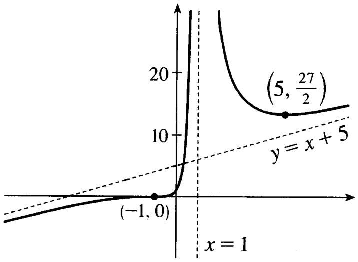
E. $f'(x) = \dfrac{(x-1)^2 \cdot 3(x+1)^2 - (x+1)^3 \cdot 2(x-1)}{[(x-1)^2]^2}$

$= \dfrac{(x-1)(x+1)^2[3(x-1) - 2(x+1)]}{(x-1)^4} = \dfrac{(x+1)^2(x-5)}{(x-1)^3}$

so $f'(x) > 0$ when $x < -1$, $-1 < x < 1$, or $x > 5$, and $f'(x) < 0$ when $1 < x < 5$. f is increasing on $(-\infty, 1)$ and $(5, \infty)$ and decreasing on $(1, 5)$.

F. Local minimum value $f(5) = \frac{216}{16} = \frac{27}{2}$, no local maximum

H.



G. $f''(x) = \dfrac{(x-1)^3[(x-1)^2 + (x-5) \cdot 2(x+1)] - (x+1)^2(x-5) \cdot 3(x-1)^2}{[(x-1)^3]^2}$

$= \dfrac{(x-1)^2(x+1)\{(x-1)[(x+1) + 2(x-5)] - 3(x+1)(x-5)\}}{(x-1)^6}$

$= \dfrac{(x+1)\{(x-1)[3x-9] - 3(x^2 - 4x - 5)\}}{(x-1)^4} = \dfrac{(x+1)(24)}{(x-1)^4}$

so $f''(x) > 0$ if $-1 < x < 1$ or $x > 1$, and $f''(x) < 0$ if $x < -1$. Thus, f is CU on $(-1, 1)$ and $(1, \infty)$ and CD on $(-\infty, -1)$. IP at $(-1, 0)$

65. $y = f(x) = x - \tan^{-1} x$, $f'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{1 + x^2 - 1}{1+x^2} = \dfrac{x^2}{1+x^2}$,

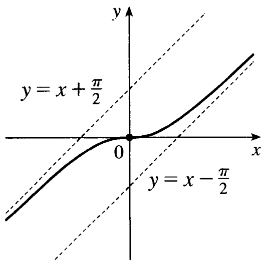
$f''(x) = \dfrac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} = \dfrac{2x(1 + x^2 - x^2)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2}$.

$\lim\limits_{x\to\infty} \left[f(x) - \left(x - \frac{\pi}{2}\right)\right] = \lim\limits_{x\to\infty} \left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{\pi}{2} - \frac{\pi}{2} = 0$, so $y = x - \frac{\pi}{2}$ is a SA. Also,

$\lim\limits_{x\to-\infty} \left[f(x) - \left(x + \frac{\pi}{2}\right)\right] = \lim\limits_{x\to-\infty} \left(-\frac{\pi}{2} - \tan^{-1} x\right)$

$= -\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = 0$

so $y = x + \frac{\pi}{2}$ is also a SA. $f'(x) \geq 0$ for all x, with equality $\Leftrightarrow$ $x = 0$, so f is increasing on $\mathbb{R}$. $f''(x)$ has the same sign as x, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. $f(-x) = -f(x)$, so f is an odd function; its graph is symmetric about the origin. f has no local extreme values. Its only IP is at $(0, 0)$.



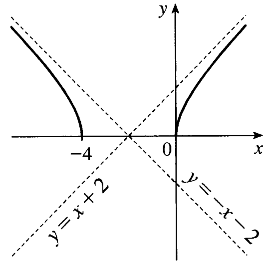
66. $y = f(x) = \sqrt{x^2 + 4x} = \sqrt{x(x+4)}$. $x(x+4) \geq 0$ $\Leftrightarrow$ $x \leq -4$ or $x \geq 0$, so $D = (-\infty, -4] \cup [0, \infty)$.

y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0$ $\Rightarrow$ $x = -4, 0$.

$\sqrt{x^2 + 4x} \mp (x+2) = \dfrac{\sqrt{x^2+4x} \mp (x+2)}{1} \cdot \dfrac{\sqrt{x^2+4x} \pm (x+2)}{\sqrt{x^2+4x} \pm (x+2)} = \dfrac{(x^2+4x) - (x^2+4x+4)}{\sqrt{x^2+4x} \pm (x+2)}$

$= \dfrac{-4}{\sqrt{x^2+4x} \pm (x+2)}$

so $\lim\limits_{x\to\pm\infty} [f(x) \mp (x+2)] = 0$. Thus, the graph of f approaches the slant asymptote $y = x + 2$ as $x \to \infty$ and it approaches the slant asymptote $y = -(x+2)$ as $x \to -\infty$. $f'(x) = \dfrac{x+2}{\sqrt{x^2+4x}}$, so $f'(x) < 0$ for $x < -4$ and $f'(x) > 0$ for $x > 0$; that is, f is decreasing on $(-\infty, -4)$ and increasing on $(0, \infty)$. There are no local extreme values.



$f'(x) = (x+2)(x^2+4x)^{-1/2} \quad \Rightarrow$

$$\begin{aligned} f''(x) &= (x+2)\cdot\left(-\tfrac{1}{2}\right)(x^2+4x)^{-3/2}\cdot(2x+4) + (x^2+4x)^{-1/2} \\ &= (x^2+4x)^{-3/2}\left[-(x+2)^2 + (x^2+4x)\right] \\ &= -4(x^2+4x)^{-3/2} < 0 \text{ on } D \end{aligned}$$

so f is CD on $(-\infty, -4)$ and $(0, \infty)$. No IP

67. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad y = \pm\dfrac{b}{a}\sqrt{x^2-a^2}$. Now

$$\lim_{x\to\infty}\left[\frac{b}{a}\sqrt{x^2-a^2} - \frac{b}{a}x\right] = \frac{b}{a}\cdot\lim_{x\to\infty}\left(\sqrt{x^2-a^2}-x\right)\frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}+x} = \frac{b}{a}\cdot\lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x} = 0,$$

which shows that $y = \dfrac{b}{a}x$ is a slant asymptote. Similarly,

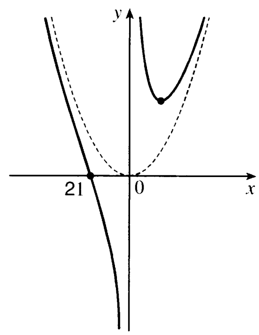
$$\lim_{x\to\infty}\left[-\frac{b}{a}\sqrt{x^2-a^2} - \left(-\frac{b}{a}x\right)\right] = -\frac{b}{a}\cdot\lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x} = 0, \text{ so } y = -\frac{b}{a}x \text{ is a slant asymptote.}$$

68. $f(x) - x^2 = \dfrac{x^3+1}{x} - x^2 = \dfrac{x^3+1-x^3}{x} = \dfrac{1}{x}$, and $\displaystyle\lim_{x\to\pm\infty}\frac{1}{x} = 0$. Therefore, $\displaystyle\lim_{x\to\pm\infty}\left[f(x)-x^2\right] = 0$, and so the graph of f is asymptotic to that of $y = x^2$. For purposes of differentiation, we will use $f(x) = x^2 + 1/x$.

A. $D = \{x \mid x \neq 0\}$ **B.** No y-intercept; to find the x-intercept, we set $y = 0 \quad \Leftrightarrow \quad x = -1$.

C. No symmetry **D.** $\displaystyle\lim_{x\to 0^+}\frac{x^3+1}{x} = \infty$ and $\displaystyle\lim_{x\to 0^-}\frac{x^3+1}{x} = -\infty$, so $x = 0$ is a vertical asymptote. Also, the graph is asymptotic to the parabola $y = x^2$, as shown above. **E.** $f'(x) = 2x - 1/x^2 > 0 \quad \Leftrightarrow$ $x > \frac{1}{\sqrt[3]{2}}$, so f is increasing on $\left(\frac{1}{\sqrt[3]{2}}, \infty\right)$ and decreasing on $(-\infty, 0)$ and $\left(0, \frac{1}{\sqrt[3]{2}}\right)$. **F.** Local minimum value $f\left(\frac{1}{\sqrt[3]{2}}\right) = \frac{3\sqrt[3]{3}}{2}$, no local maximum **G.** $f''(x) = 2 + 2/x^3 > 0 \quad \Leftrightarrow \quad x < -1$ or $x > 0$, so f is CU on $(-\infty, -1)$ and $(0, \infty)$, and CD on $(-1, 0)$. IP at $(-1, 0)$

H.

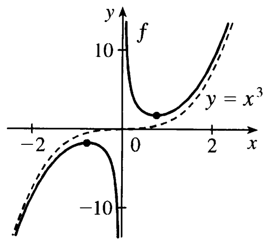


69. $\displaystyle\lim_{x\to\pm\infty}\left[f(x)-x^3\right] = \lim_{x\to\pm\infty}\frac{x^4+1}{x} - \frac{x^4}{x} = \lim_{x\to\pm\infty}\frac{1}{x} = 0$, so the graph of f is asymptotic to that of $y = x^3$.

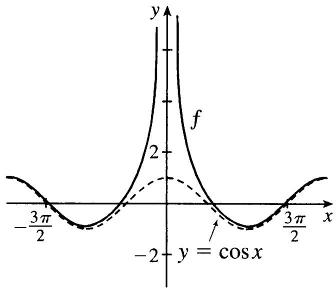
A. $D = \{x \mid x \neq 0\}$ **B.** No intercept **C.** f is symmetric about the origin. **D.** $\displaystyle\lim_{x\to 0^-}\left(x^3 + \frac{1}{x}\right) = -\infty$ and $\displaystyle\lim_{x\to 0^+}\left(x^3 + \frac{1}{x}\right) = \infty$, so $x = 0$ is a vertical asymptote, and as shown above, the graph of f is asymptotic to

that of $y = x^3$. **E.** $f'(x) = 3x^2 - 1/x^2 > 0 \Leftrightarrow x^4 > \frac{1}{3} \Leftrightarrow$ $|x| > \frac{1}{\sqrt[4]{3}}$, so f is increasing on $\left(-\infty, -\frac{1}{\sqrt[4]{3}}\right)$ and $\left(\frac{1}{\sqrt[4]{3}}, \infty\right)$ and decreasing on $\left(-\frac{1}{\sqrt[4]{3}}, 0\right)$ and $\left(0, \frac{1}{\sqrt[4]{3}}\right)$. **F.** Local maximum value $f\left(-\frac{1}{\sqrt[4]{3}}\right) = -4 \cdot 3^{-5/4}$, local minimum value $f\left(\frac{1}{\sqrt[4]{3}}\right) = 4 \cdot 3^{-5/4}$
G. $f''(x) = 6x + 2/x^3 > 0 \Leftrightarrow x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

H.

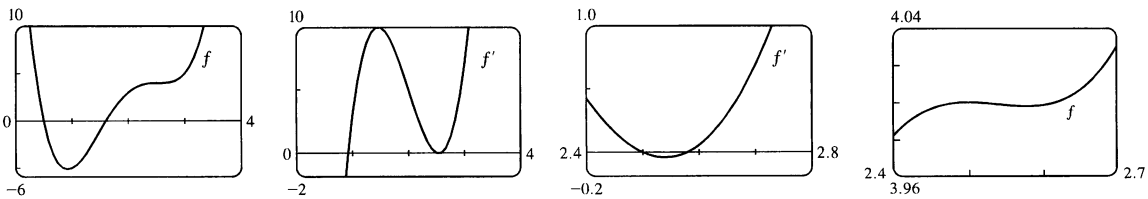


70. $\lim\limits_{x \to \pm\infty} [f(x) - \cos x] = \lim\limits_{x \to \pm\infty} 1/x^2 = 0$, so the graph of f is asymptotic to that of $\cos x$. The intercepts can only be found approximately. $f(x) = f(-x)$, so f is even. $\lim\limits_{x \to 0}\left(\cos x + \frac{1}{x^2}\right) = \infty$, so $x = 0$ is a vertical asymptote. We don't need to calculate the derivatives, since we know the asymptotic behavior of the curve.



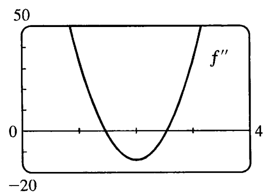
4.6 Graphing with Calculus and Calculators

1. $f(x) = 4x^4 - 32x^3 + 89x^2 - 95x + 29 \Rightarrow f'(x) = 16x^3 - 96x^2 + 178x - 95 \Rightarrow$ $f''(x) = 48x^2 - 192x + 178$. $f(x) = 0 \Leftrightarrow x \approx 0.5, 1.60$; $f'(x) = 0 \Leftrightarrow x \approx 0.92, 2.5, 2.58$ and $f''(x) = 0 \Leftrightarrow x \approx 1.46, 2.54$.



From the graphs of f', we estimate that $f' < 0$ and that f is decreasing on $(-\infty, 0.92)$ and $(2.5, 2.58)$, and that $f' > 0$ and f is increasing on $(0.92, 2.5)$ and $(2.58, \infty)$ with local minimum values $f(0.92) \approx -5.12$ and $f(2.58) \approx 3.998$ and local maximum value $f(2.5) = 4$. The graphs of f' make it clear that f has a maximum and a minimum near $x = 2.5$, shown more clearly in the fourth graph.

From the graph of f'', we estimate that $f'' > 0$ and that f is CU on $(-\infty, 1.46)$ and $(2.54, \infty)$, and that $f'' < 0$ and f is CD on $(1.46, 2.54)$. There are inflection points at about $(1.46, -1.40)$ and $(2.54, 3.999)$.

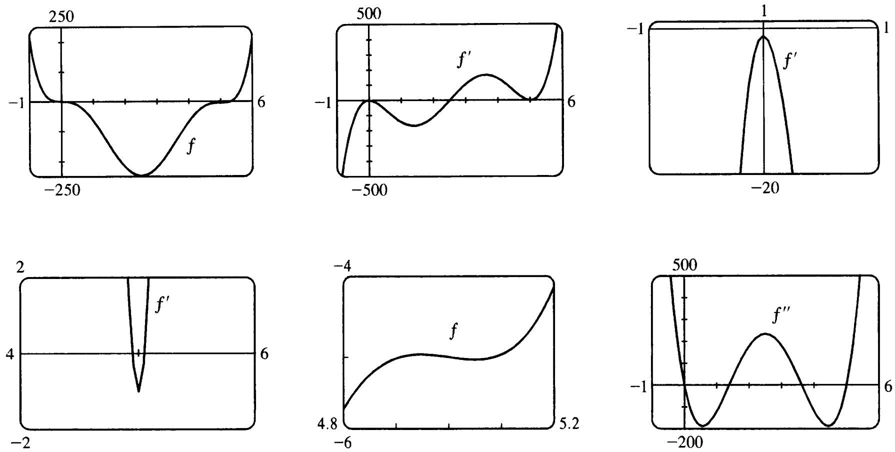


2. $f(x) = x^6 - 15x^5 + 75x^4 - 125x^3 - x \quad \Rightarrow$

$f'(x) = 6x^5 - 75x^4 + 300x^3 - 375x^2 - 1 \quad \Rightarrow \quad f''(x) = 30x^4 - 300x^3 + 900x^2 - 750x.$

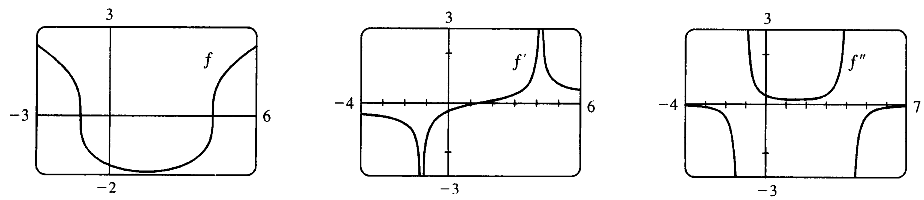
$f(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $x \approx 5.33$; $\quad f'(x) = 0 \quad \Leftrightarrow \quad x \approx 2.50, 4.95,$ or 5.05;

$f''(x) = 0 \quad \Leftrightarrow \quad x = 0, 5$ or $x \approx 1.38, 3.62$.



From the graphs of f', we estimate that f is decreasing on $(-\infty, 2.50)$, increasing on $(2.50, 4.95)$, decreasing on $(4.95, 5.05)$, and increasing on $(5.05, \infty)$, with local minimum values $f(2.50) \approx -246.6$ and $f(5.05) \approx -5.03$, and local maximum value $f(4.95) \approx -4.965$ (notice the second graph of f). From the graph of f'', we estimate that f is CU on $(-\infty, 0)$, CD on $(0, 1.38)$, CU on $(1.38, 3.62)$, CD on $(3.62, 5)$, and CU on $(5, \infty)$. There are inflection points at $(0, 0)$ and $(5, -5)$, and at about $(1.38, -126.38)$ and $(3.62, -128.62)$.

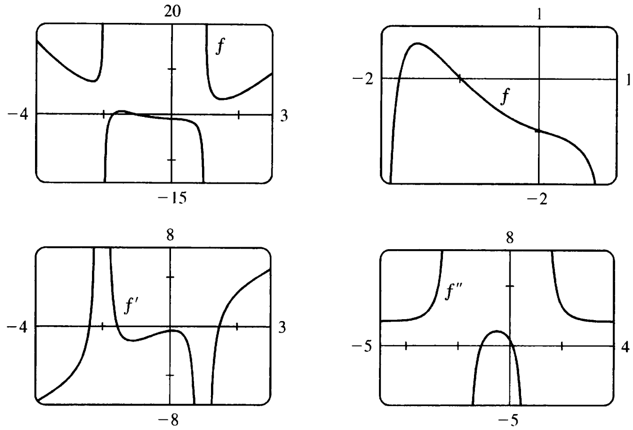
3. $f(x) = \sqrt[3]{x^2 - 3x - 5} \quad \Rightarrow \quad f'(x) = \dfrac{1}{3}\dfrac{2x - 3}{(x^2 - 3x - 5)^{2/3}} \quad \Rightarrow \quad f''(x) = -\dfrac{2}{9}\dfrac{x^2 - 3x + 24}{(x^2 - 3x - 5)^{5/3}}$



Note: With some CAS's, including Maple, it is necessary to define $f(x) = \dfrac{x^2 - 3x - 5}{|x^2 - 3x - 5|}\left|x^2 - 3x - 5\right|^{1/3}$, since the CAS does not compute real cube roots of negative numbers. We estimate from the graph of f' that f is increasing on $(1.5, \infty)$, and decreasing on $(-\infty, 1.5)$. f has no maximum. Minimum value: $f(1.5) \approx -1.9$. From the graph of f'', we estimate that f is CU on $(-1.2, 4.2)$ and CD on $(-\infty, -1.2)$ and $(4.2, \infty)$. IP at $(-1.2, 0)$ and $(4.2, 0)$.

4. $f(x) = \dfrac{x^4 + x^3 - 2x^2 + 2}{x^2 + x - 2} \quad \Rightarrow \quad f'(x) = 2\,\dfrac{x^5 + 2x^4 - 3x^3 - 4x^2 + 2x - 1}{(x^2 + x - 2)^2} \quad \Rightarrow$

$$f''(x) = 2\,\frac{x^6 + 3x^5 - 3x^4 - 11x^3 + 12x^2 + 18x - 2}{(x^2 + x - 2)^3}$$



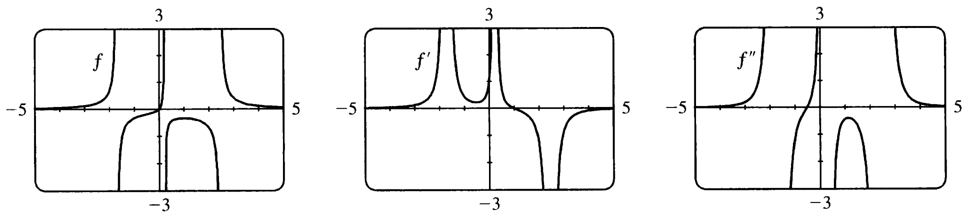
We estimate from the graph of f' that f is increasing on $(-2.4, -2)$, $(-2, -1.5)$ and $(1.5, \infty)$ and decreasing on $(-\infty, -2.4)$, $(-1.5, 1)$ and $(1, 1.5)$. Local maximum value: $f(-1.5) \approx 0.7$.

Local minimum values: $f(-2.4) \approx 7.2$, $f(1.5) \approx 3.4$. From the graph of f'', we estimate that f is CU on $(-\infty, -2)$, $(-1.1, 0.1)$ and $(1, \infty)$ and CD on $(-2, -1.1)$ and $(0.1, 1)$.

f has IP at $(-1.1, 0.2)$ and $(0.1, -1.1)$.

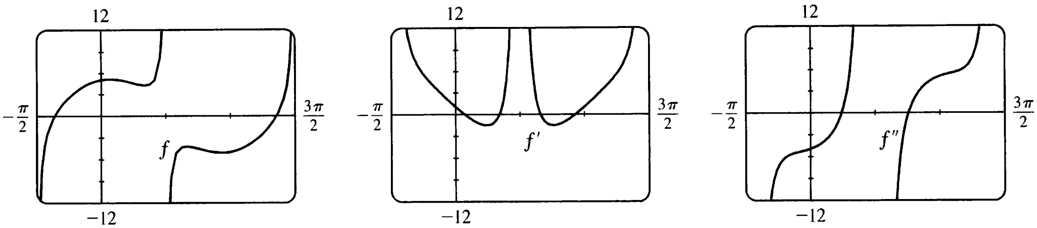
5. $f(x) = \dfrac{x}{x^3 - x^2 - 4x + 1} \quad \Rightarrow \quad f'(x) = \dfrac{-2x^3 + x^2 + 1}{(x^3 - x^2 - 4x + 1)^2} \quad \Rightarrow$

$$f''(x) = \frac{2(3x^5 - 3x^4 + 5x^3 - 6x^2 + 3x + 4)}{(x^3 - x^2 - 4x + 1)^3}$$



We estimate from the graph of f that $y = 0$ is a horizontal asymptote, and that there are vertical asymptotes at $x = -1.7$, $x = 0.24$, and $x = 2.46$. From the graph of f', we estimate that f is increasing on $(-\infty, -1.7)$, $(-1.7, 0.24)$, and $(0.24, 1)$, and that f is decreasing on $(1, 2.46)$ and $(2.46, \infty)$. There is a local maximum value at $f(1) = -\frac{1}{3}$. From the graph of f'', we estimate that f is CU on $(-\infty, -1.7)$, $(-0.506, 0.24)$, and $(2.46, \infty)$, and that f is CD on $(-1.7, -0.506)$ and $(0.24, 2.46)$. There is an inflection point at $(-0.506, -0.192)$.

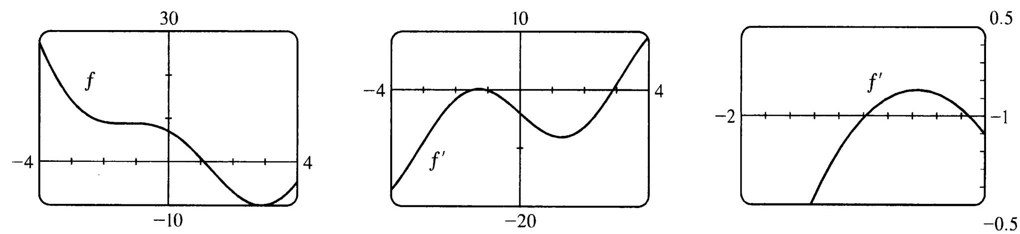
6. $f(x) = \tan x + 5\cos x \quad \Rightarrow \quad f'(x) = \sec^2 x - 5\sin x \quad \Rightarrow \quad f''(x) = 2\sec^2 x\,\tan x - 5\,\cos x$. Since f is periodic with period 2π, and defined for all x except odd multiples of $\frac{\pi}{2}$, we graph f and its derivatives on $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

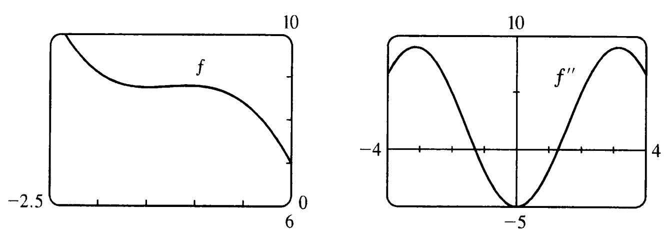


We estimate from the graph of f' that f is increasing on $\left(-\frac{\pi}{2}, 0.21\right)$, $\left(1.07, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, 2.07\right)$, and $\left(2.93, \frac{3\pi}{2}\right)$, and decreasing on $(0.21, 1.07)$ and $(2.07, 2.93)$. Local minimum values: $f(1.07) \approx 4.23$, $f(2.93) \approx -5.10$. Local maximum values: $f(0.21) \approx 5.10$, $f(2.07) \approx -4.23$.

From the graph of f'', we estimate that f is CU on $\left(0.76, \frac{\pi}{2}\right)$ and $\left(2.38, \frac{3\pi}{2}\right)$, and CD on $\left(-\frac{\pi}{2}, 0.76\right)$ and $\left(\frac{\pi}{2}, 2.38\right)$. f has IP at $(0.76, 4.57)$ and $(2.38, -4.57)$.

7. $f(x) = x^2 - 4x + 7\cos x$, $-4 \le x \le 4$. $f'(x) = 2x - 4 - 7\sin x \quad \Rightarrow \quad f''(x) = 2 - 7\cos x$.
$f(x) = 0 \quad \Leftrightarrow \quad x \approx 1.10$; $f'(x) = 0 \quad \Leftrightarrow \quad x \approx -1.49, -1.07$, or 2.89; $f''(x) = 0 \quad \Leftrightarrow$
$x = \pm\cos^{-1}\left(\frac{2}{7}\right) \approx \pm 1.28$.





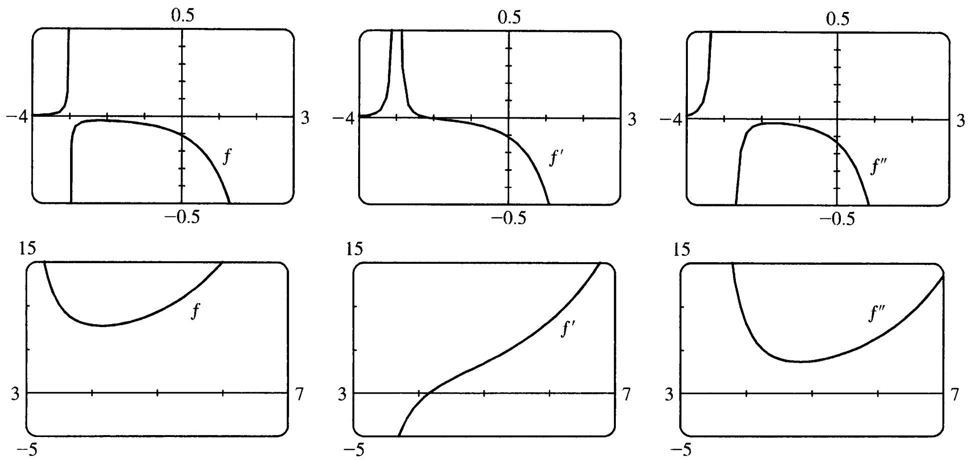
From the graphs of f', we estimate that f is decreasing $(f' < 0)$ on $(-4, -1.49)$, increasing on $(-1.49, -1.07)$, decreasing on $(-1.07, 2.89)$, and increasing on $(2.89, 4)$, with local minimum values $f(-1.49) \approx 8.75$ and $f(2.89) \approx -9.99$ and local maximum value $f(-1.07) \approx 8.79$ (notice the second graph of f). From the graph of f'', we estimate that f is CU ($f'' > 0$) on $(-4, -1.28)$, CD on $(-1.28, 1.28)$, and CU on $(1.28, 4)$. There are inflection points at about $(-1.28, 8.77)$ and $(1.28, -1.48)$.

8. $f(x) = \dfrac{e^x}{x^2 - 9} \quad \Rightarrow \quad f'(x) = \dfrac{e^x(x^2 - 2x - 9)}{(x^2 - 9)^2} \quad \Rightarrow \quad f''(x) = \dfrac{e^x(x^4 - 4x^3 - 12x^2 + 36x + 99)}{(x^2 - 9)^3}$

There are vertical asymptotes at $x = \pm 3$. It is difficult to show all the pertinent features in one viewing rectangle, so we'll show f, f', and f'' for $x < 3$ and also for $x > 3$.

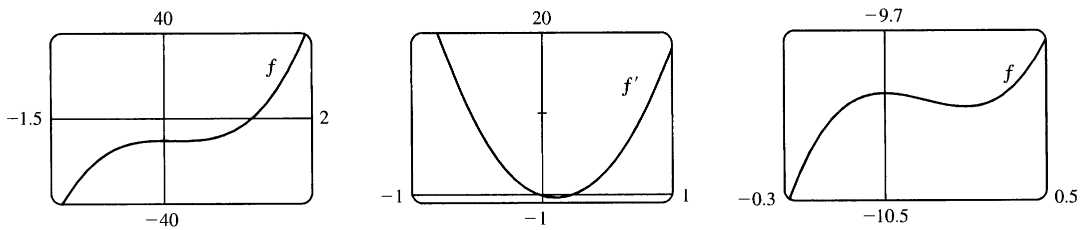
For $x < 3$:

For $x > 3$:



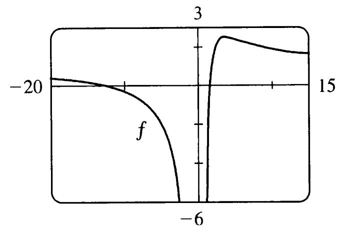
We estimate from the graphs of f' and f that f is increasing on $(-\infty, -3)$, $(-3, -2.16)$, and $(4.16, \infty)$ and decreasing on $(-2.16, 3)$ and $(3, 4.16)$. There is a local maximum value of $f(-2.16) \approx -0.03$ and a local minimum value of $f(4.16) \approx 7.71$. From the graphs of f'', we see that f is CU on $(-\infty, -3)$ and $(3, \infty)$ and CD on $(-3, 3)$. There is no inflection point.

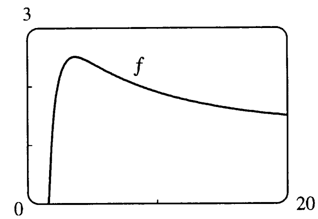
9. $f(x) = 8x^3 - 3x^2 - 10 \quad \Rightarrow \quad f'(x) = 24x^2 - 6x \quad \Rightarrow \quad f''(x) = 48x - 6$



From the graphs, it appears that $f(x) = 8x^3 - 3x^2 - 10$ increases on $(-\infty, 0)$ and $(0.25, \infty)$ and decreases on $(0, 0.25)$; that f has a local maximum value of $f(0) = -10.0$ and a local minimum value of $f(0.25) \approx -10.1$; that f is CU on $(0.1, \infty)$ and CD on $(-\infty, 0.1)$; and that f has an IP at $(0.1, -10)$. To find the exact values, note that $f'(x) = 24x^2 - 6x = 6x(4x - 1)$, which is positive ($f$ is increasing) for $(-\infty, 0)$ and $\left(\frac{1}{4}, \infty\right)$, and negative ($f$ is decreasing) on $\left(0, \frac{1}{4}\right)$. By the FDT, f has a local maximum at $x = 0$: $f(0) = -10$; and f has a local minimum at $\frac{1}{4}$: $f\left(\frac{1}{4}\right) = \frac{1}{8} - \frac{3}{16} - 10 = -\frac{161}{16}$. $f''(x) = 48x - 6 = 6(8x - 1)$, which is positive ($f$ is CU) on $\left(\frac{1}{8}, \infty\right)$ and negative (f is CD) on $\left(-\infty, \frac{1}{8}\right)$. f has an IP at $\left(\frac{1}{8}, f\left(\frac{1}{8}\right)\right) = \left(\frac{1}{8}, -\frac{321}{32}\right)$.

10.





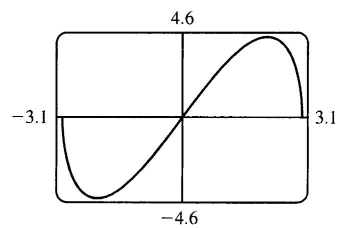
From the graphs, it appears that f increases on $(0, 3.6)$ and decreases on $(-\infty, 0)$ and $(3.6, \infty)$; that f has a local maximum of $f(3.6) \approx 2.5$ and no local minima; that f is CU on $(5.5, \infty)$ and CD on $(-\infty, 0)$ and $(0, 5.5)$; and that f has an IP at $(5.5, 2.3)$. $f(x) = \dfrac{x^2 + 11x - 20}{x^2} = 1 + \dfrac{11}{x} - \dfrac{20}{x^2} \Rightarrow$

$f'(x) = -11x^{-2} + 40x^{-3} = -x^{-3}(11x - 40)$, which is positive ($f$ is increasing) on $\left(0, \frac{40}{11}\right)$, and negative ($f$ is decreasing) on $(-\infty, 0)$ and on $\left(\frac{40}{11}, \infty\right)$. By the FDT, f has a local maximum at $x = \frac{40}{11}$:

$$f\left(\tfrac{40}{11}\right) = \frac{\left(\frac{40}{11}\right)^2 + 11\left(\frac{40}{11}\right) - 20}{\left(\frac{40}{11}\right)^2} = \frac{1600 + 11 \cdot 11 \cdot 40 - 20 \cdot 121}{1600} = \frac{201}{80}$$; and f has no local minimum.

$f'(x) = -11x^{-2} + 40x^{-3} \Rightarrow f''(x) = 22x^{-3} - 120x^{-4} = 2x^{-4}(11x - 60)$, which is positive ($f$ is CU) on $\left(\frac{60}{11}, \infty\right)$, and negative ($f$ is CD) on $(-\infty, 0)$ and $\left(0, \frac{60}{11}\right)$. f has an IP at $\left(\frac{60}{11}, f\left(\frac{60}{11}\right)\right) = \left(\frac{60}{11}, \frac{211}{90}\right)$.

11. From the graph, it appears that f increases on $(-2.1, 2.1)$ and decreases on $(-3, -2.1)$ and $(2.1, 3)$; that f has a local maximum of $f(2.1) \approx 4.5$ and a local minimum of $f(-2.1) \approx -4.5$; that f is CU on $(-3.0, 0)$ and CD on $(0, 3.0)$, and that f has an IP at $(0, 0)$. $f(x) = x\sqrt{9 - x^2} \Rightarrow$



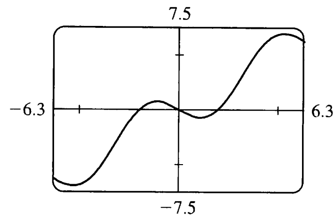
$f'(x) = \dfrac{-x^2}{\sqrt{9 - x^2}} + \sqrt{9 - x^2} = \dfrac{9 - 2x^2}{\sqrt{9 - x^2}}$, which is positive ($f$ is increasing) on $\left(\frac{-3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ and negative (f is decreasing) on $\left(-3, \frac{-3\sqrt{2}}{2}\right)$ and $\left(\frac{3\sqrt{2}}{2}, 3\right)$. By the FDT, f has a local maximum value of $f\left(\frac{3\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}\sqrt{9 - \left(\frac{3\sqrt{2}}{2}\right)^2} = \frac{9}{2}$; and f has a local minimum value of $f\left(\frac{-3\sqrt{2}}{2}\right) = -\frac{9}{2}$ (since f is an odd function). $f'(x) = \dfrac{-x^2}{\sqrt{9 - x^2}} + \sqrt{9 - x^2} \Rightarrow$

$$f''(x) = \frac{\sqrt{9 - x^2}\,(-2x) + x^2\left(\frac{1}{2}\right)(9 - x^2)^{-1/2}(-2x)}{9 - x^2} - x(9 - x^2)^{-1/2} = \frac{-2x - x^3(9 - x^2)^{-1} - x}{\sqrt{9 - x^2}}$$

$$= \frac{-3x}{\sqrt{9 - x^2}} - \frac{x^3}{(9 - x^2)^{3/2}} = \frac{x(2x^2 - 27)}{(9 - x^2)^{3/2}}$$

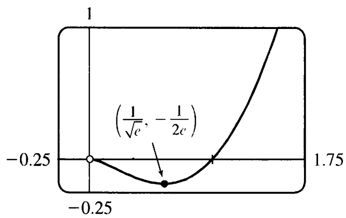
which is positive (f is CU) on $(-3, 0)$ and negative (f is CD) on $(0, 3)$. f has an IP at $(0, 0)$.

12. From the graph, it appears that f increases on $(-5.2, -1.0)$ and $(1.0, 5.2)$ and decreases on $(-2\pi, -5.2)$, $(-1.0, 1.0)$, and $(5.2, 2\pi)$; that f has local maximum values of $f(-1.0) \approx 0.7$ and $f(5.2) \approx 7.0$ and local minimum values of $f(-5.2) \approx -7.0$ and $f(1.0) \approx -0.7$; that f is CU on $(-2\pi, -3.1)$ and $(0, 3.1)$ and CD on $(-3.1, 0)$ and $(3.1, 2\pi)$, and that f has IP at $(0,0)$, $(-3.1, -3.1)$ and $(3.1, 3.1)$. $f(x) = x - 2\sin x \;\Rightarrow$



$f'(x) = 1 - 2\cos x$, which is positive (f is increasing) when $\cos x < \frac{1}{2}$, that is, on $\left(-\frac{5\pi}{3}, -\frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$, and negative ($f$ is decreasing) on $\left(-2\pi, -\frac{5\pi}{3}\right)$, $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, and $\left(\frac{5\pi}{3}, 2\pi\right)$. By the FDT, f has local maximum values of $f\left(-\frac{\pi}{3}\right) = \frac{\pi}{3} + \sqrt{3}$ and $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$, and local minimum values of $f\left(-\frac{5\pi}{3}\right) = -\frac{5\pi}{3} - \sqrt{3}$ and $f\left(\frac{\pi}{3}\right) = -\frac{\pi}{3} - \sqrt{3}$. $f'(x) = 1 - 2\cos x \;\Rightarrow\; f''(x) = 2\sin x$, which is positive ($f$ is CU) on $(-2\pi, -\pi)$ and $(0, \pi)$ and negative (f is CD) on $(-\pi, 0)$ and $(\pi, 2\pi)$. f has IP at $(0,0)$, $(-\pi, -\pi)$ and (π, π).

13. (a) $f(x) = x^2 \ln x$. The domain of f is $(0, \infty)$.

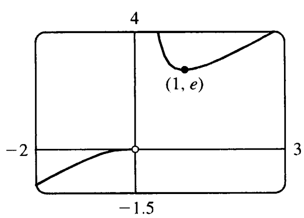


(b) $\displaystyle\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+}\left(-\frac{x^2}{2}\right) = 0$. There is a hole at $(0,0)$.

(c) It appears that there is an IP at about $(0.2, -0.06)$ and a local minimum at $(0.6, -0.18)$. $f(x) = x^2 \ln x \;\Rightarrow$ $f'(x) = x^2(1/x) + (\ln x)(2x) = x(2\ln x + 1) > 0 \;\Leftrightarrow\; \ln x > -\frac{1}{2} \;\Leftrightarrow\; x > e^{-1/2}$, so f is increasing on $(1/\sqrt{e}, \infty)$, decreasing on $(0, 1/\sqrt{e})$. By the FDT, $f(1/\sqrt{e}) = -1/(2e)$ is a local minimum value. This point is approximately $(0.6065, -0.1839)$, which agrees with our estimate.

$f''(x) = x(2/x) + (2\ln x + 1) = 2\ln x + 3 > 0 \;\Leftrightarrow\; \ln x > -\frac{3}{2} \;\Leftrightarrow\; x > e^{-3/2}$, so f is CU on $\left(e^{-3/2}, \infty\right)$ and CD on $\left(0, e^{-3/2}\right)$. IP is $\left(e^{-3/2}, -3/(2e^3)\right) \approx (0.2231, -0.0747)$.

14. (a) $f(x) = xe^{1/x}$. The domain of f is $(-\infty, 0) \cup (0, \infty)$.



(b) $\displaystyle\lim_{x\to 0^+} xe^{1/x} = \lim_{x\to 0^+} \frac{e^{1/x}}{1/x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{e^{1/x}(-1/x^2)}{-1/x^2} = \lim_{x\to 0^+} e^{1/x} = \infty$, so $x = 0$ is a VA.

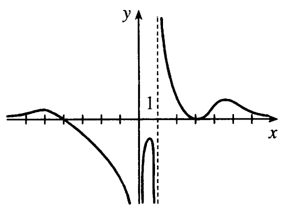
Also $\displaystyle\lim_{x\to 0^-} xe^{1/x} = 0$ since $1/x \to -\infty \;\Rightarrow\; e^{1/x} \to 0$.

(c) It appears that there is a local minimum at $(1, 2.7)$. There are no IP and f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$.

$f(x) = xe^{1/x} \;\Rightarrow\; f'(x) = xe^{1/x}\left(-\dfrac{1}{x^2}\right) + e^{1/x} = e^{1/x}\left(1 - \dfrac{1}{x}\right) > 0 \;\Leftrightarrow\; \dfrac{1}{x} < 1 \;\Leftrightarrow\; x < 0$ or $x > 1$, so f is increasing on $(-\infty, 0)$ and $(1, \infty)$, and decreasing on $(0, 1)$. By the FDT, $f(1) = e$ is a local minimum value, which agrees with our estimate.

$f''(x) = e^{1/x}(1/x^2) + (1 - 1/x)e^{1/x}(-1/x^2) = \left(e^{1/x}/x^2\right)(1 - 1 + 1/x) = e^{1/x}/x^3 > 0 \;\Leftrightarrow\; x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP.

15.

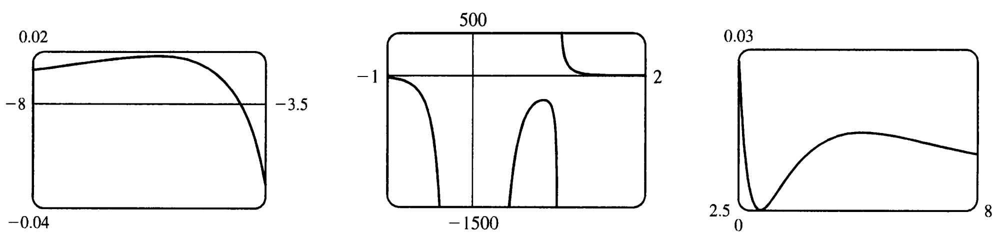


$f(x) = \dfrac{(x+4)(x-3)^2}{x^4(x-1)}$ has VA at $x = 0$ and at $x = 1$ since $\lim\limits_{x\to 0} f(x) = -\infty$, $\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$.

$$f(x) = \frac{\dfrac{x+4}{x} \cdot \dfrac{(x-3)^2}{x^2}}{\dfrac{x^4}{x^3} \cdot (x-1)} \quad \text{[dividing numerator and denominator by } x^3\text{]}$$

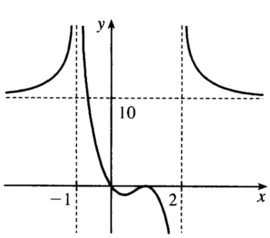
$$= \frac{(1 + 4/x)(1 - 3/x)^2}{x(x-1)} \to 0 \text{ as } x \to \pm\infty, \text{ so } f \text{ is asymptotic}$$

to the x-axis. Since f is undefined at $x = 0$, it has no y-intercept. $f(x) = 0 \;\Rightarrow\; (x+4)(x-3)^2 = 0 \;\Rightarrow\; x = -4$ or $x = 3$, so f has x-intercepts -4 and 3. Note, however, that the graph of f is only tangent to the x-axis and does not cross it at $x = 3$, since f is positive as $x \to 3^-$ and as $x \to 3^+$.



From these graphs, it appears that f has three maximum values and one minimum value. The maximum values are approximately $f(-5.6) = 0.0182$, $f(0.82) = -281.5$ and $f(5.2) = 0.0145$ and we know (since the graph is tangent to the x-axis at $x = 3$) that the minimum value is $f(3) = 0$.

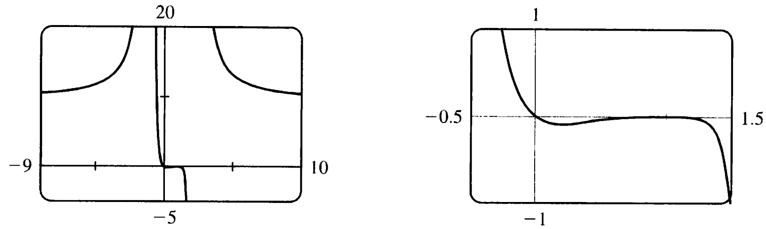
16.



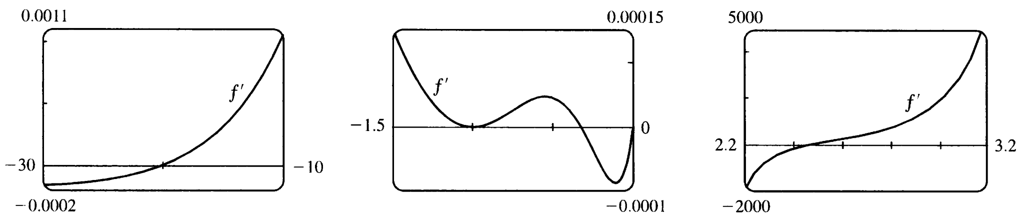
$f(x) = \dfrac{10x(x-1)^4}{(x-2)^3(x+1)^2}$ has VA at $x = -1$ and at $x = 2$ since $\lim\limits_{x\to -1} f(x) = \infty$, $\lim\limits_{x\to 2^-} f(x) = -\infty$ and $\lim\limits_{x\to 2^+} f(x) = \infty$.

$f(x) = \dfrac{10(1 - 1/x)^4}{(1 - 2/x)^3(1 + 1/x)^2} \to 10$ as $x \to \pm\infty$, so f is asymptotic to the line $y = 10$. $f(0) = 0$, so f has a y-intercept at 0. $f(x) = 0 \;\Rightarrow\; 10x(x-1)^4 = 0 \;\Rightarrow\; x = 0$ or $x = 1$. So f has x-intercepts 0 and 1.

Note, however, that f does not change sign at $x = 1$, so the graph is tangent to the x-axis and does not cross it. We know (since the graph is tangent to the x-axis at $x = 1$) that the maximum value is $f(1) = 0$. From the graphs it appears that the minimum value is about $f(0.2) = -0.1$.

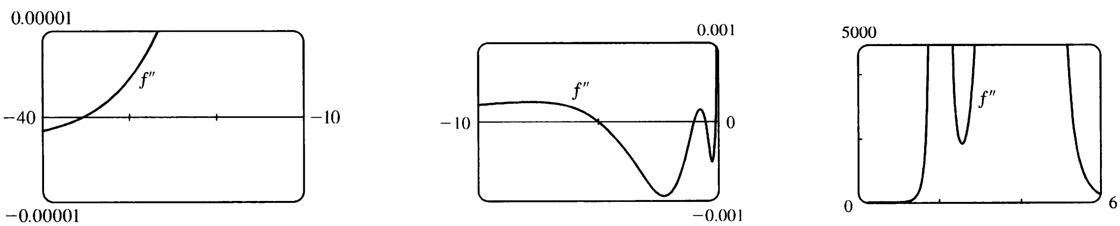


17. $f(x) = \dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4} \quad \Rightarrow \quad f'(x) = -\dfrac{x(x+1)^2(x^3+18x^2-44x-16)}{(x-2)^3(x-4)^5}$ (from CAS).



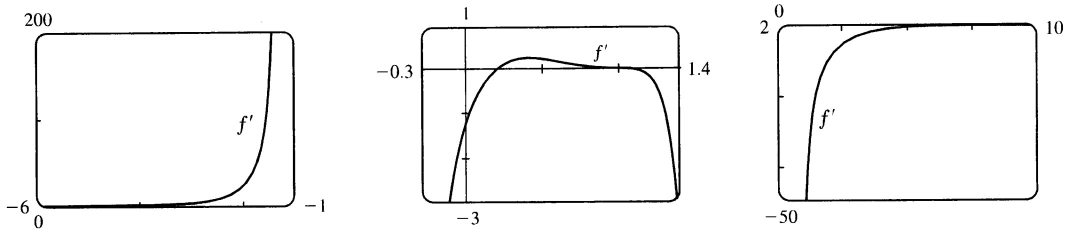
From the graphs of f', it seems that the critical points which indicate extrema occur at $x \approx -20$, -0.3, and 2.5, as estimated in Example 3. (There is another critical point at $x = -1$, but the sign of f' does not change there.)

We differentiate again, obtaining $f''(x) = 2\dfrac{(x+1)(x^6+36x^5+6x^4-628x^3+684x^2+672x+64)}{(x-2)^4(x-4)^6}$.

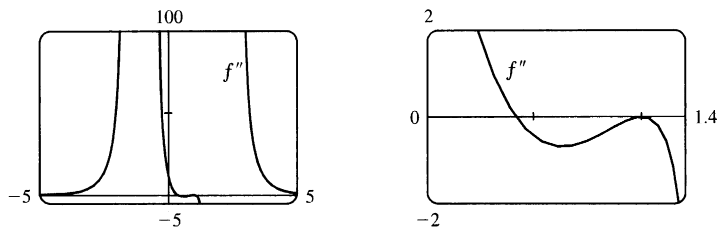


From the graphs of f'', it appears that f is CU on $(-35.3, -5.0)$, $(-1, -0.5)$, $(-0.1, 2)$, $(2, 4)$ and $(4, \infty)$ and CD on $(-\infty, -35.3)$, $(-5.0, -1)$ and $(-0.5, -0.1)$. We check back on the graphs of f to find the y-coordinates of the inflection points, and find that these points are approximately $(-35.3, -0.015)$, $(-5.0, -0.005)$, $(-1, 0)$, $(-0.5, 0.00001)$, and $(-0.1, 0.0000066)$.

18. $f(x) = \dfrac{10x(x-1)^4}{(x-2)^3(x+1)^2} \quad \Rightarrow \quad f'(x) = -20\dfrac{(x-1)^3(5x-1)}{(x-2)^4(x+1)^3}$ (from CAS).



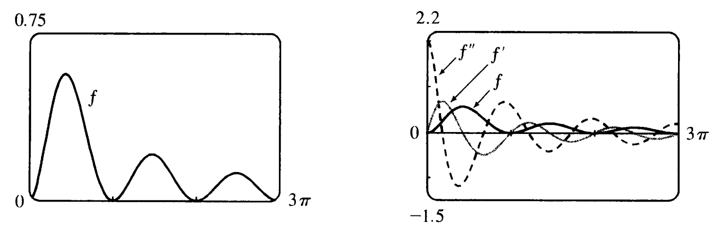
From the graphs of f', we estimate that f is increasing on $(-\infty, -1)$ and $(0.2, 1)$ and decreasing on $(-1, 0.2)$, $(1, 2)$ and $(2, \infty)$. Differentiating $f'(x)$, we get $f''(x) = 60\dfrac{(x-1)^2(5x^3 - 8x^2 + 17x - 6)}{(x-2)^5(x+1)^4}$.



From the graphs of f'', it seems that f is CU on $(-\infty, -1.0)$, $(-1.0, 0.4)$ and $(2.0, \infty)$, and CD on $(0.4, 2)$. There is an inflection point at about $(0.4, -0.06)$.

19. $y = f(x) = \dfrac{\sin^2 x}{\sqrt{x^2+1}}$ with $0 \le x \le 3\pi$. From a CAS, $y' = \dfrac{\sin x\left[2\left(x^2+1\right)\cos x - x\sin x\right]}{\left(x^2+1\right)^{3/2}}$ and

$$y'' = \frac{\left(4x^4 + 6x^2 + 5\right)\cos^2 x - 4x\left(x^2+1\right)\sin x\cos x - 2x^4 - 2x^2 - 3}{\left(x^2+1\right)^{5/2}}.$$

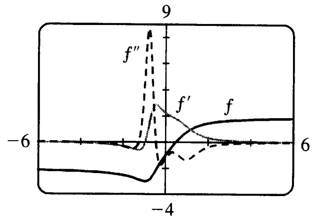


From the graph of f' and the formula for y', we determine that $y' = 0$ when $x = \pi, 2\pi, 3\pi$, or $x \approx 1.3, 4.6$, or 7.8. So f is increasing on $(0, 1.3)$, $(\pi, 4.6)$, and $(2\pi, 7.8)$. f is decreasing on $(1.3, \pi)$, $(4.6, 2\pi)$, and $(7.8, 3\pi)$. Local maximum values: $f(1.3) \approx 0.6$, $f(4.6) \approx 0.21$, and $f(7.8) \approx 0.13$. Local minimum values: $f(\pi) = f(2\pi) = 0$. From the graph of f'', we see that $y'' = 0 \quad \Leftrightarrow \quad x \approx 0.6, 2.1, 3.8, 5.4, 7.0$, or 8.6. So f is CU on $(0, 0.6)$, $(2.1, 3.8)$, $(5.4, 7.0)$, and $(8.6, 3\pi)$. f is CD on $(0.6, 2.1)$, $(3.8, 5.4)$, and $(7.0, 8.6)$. There are IP at $(0.6, 0.25)$, $(2.1, 0.31)$, $(3.8, 0.10)$, $(5.4, 0.11)$, $(7.0, 0.061)$, and $(8.6, 0.065)$.

20. $f(x) = \dfrac{2x-1}{\sqrt[4]{x^4+x+1}} \quad \Rightarrow$

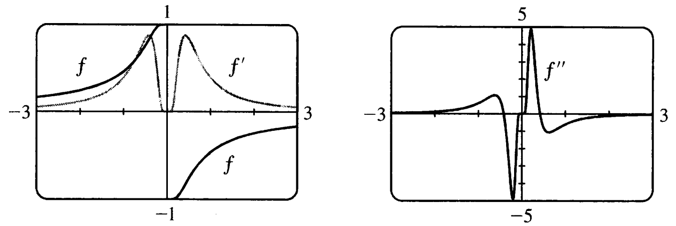
$f'(x) = \dfrac{4x^3+6x+9}{4(x^4+x+1)^{5/4}} \quad \Rightarrow$

$f''(x) = -\dfrac{32x^6+96x^4+152x^3-48x^2+6x+21}{16(x^4+x+1)^{9/4}}$



From the graph of f', f appears to be decreasing on $(-\infty, -0.94)$ and increasing on $(-0.94, \infty)$. There is a local minimum value of $f(-0.94) \approx -3.01$. From the graph of f'', f appears to be CU on $(-1.25, -0.44)$ and CD on $(-\infty, -1.25)$ and $(-0.44, \infty)$. There are inflection points at $(-1.25, -2.87)$ and $(-0.44, -2.14)$.

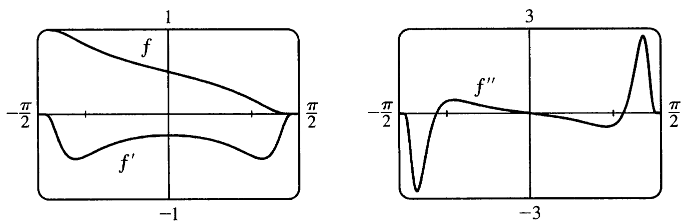
21. $y = f(x) = \dfrac{1-e^{1/x}}{1+e^{1/x}}$. From a CAS, $y' = \dfrac{2e^{1/x}}{x^2(1+e^{1/x})^2}$ and $y'' = \dfrac{-2e^{1/x}(1-e^{1/x}+2x+2xe^{1/x})}{x^4(1+e^{1/x})^3}$.



f is an odd function defined on $(-\infty, 0) \cup (0, \infty)$. Its graph has no x- or y-intercepts. Since $\lim\limits_{x\to\pm\infty} f(x) = 0$, the x-axis is a HA. $f'(x) > 0$ for $x \neq 0$, so f is increasing on $(-\infty, 0)$ and $(0, \infty)$. It has no local extreme values. $f''(x) = 0$ for $x \approx \pm 0.417$, so f is CU on $(-\infty, -0.417)$, CD on $(-0.417, 0)$, CU on $(0, 0.417)$, and CD on $(0.417, \infty)$. f has IPs at $(-0.417, 0.834)$ and $(0.417, -0.834)$.

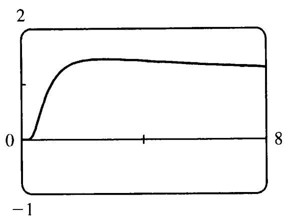
22. $y = f(x) = \dfrac{1}{1+e^{\tan x}}$. From a CAS, $y' = -\dfrac{e^{\tan x}}{\cos^2 x\,(1+e^{\tan x})^2}$ and

$y'' = -\dfrac{e^{\tan x}[e^{\tan x}(2\sin x\cos x - 1) + 2\sin x\cos x + 1]}{\cos^4 x\,(1+e^{\tan x})^3}$. f is a periodic function with period π that has positive values throughout its domain, which consists of all real numbers except odd multiples of $\frac{\pi}{2}$ (that is, $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$, and so on). f has y-intercept $\frac{1}{2}$, but no x-intercepts. We graph f, f', and f'' on one period, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



Since $f'(x) < 0$ for all x in the domain of f, f is decreasing on the intervals between odd multiples of $\frac{\pi}{2}$. $f''(x) = 0$ for $x = 0 + n\pi$ and for $x \approx \pm 1.124 + n\pi$, so f is CD on $\left(-\frac{\pi}{2}, -1.124\right)$, CU on $(-1.124, 0)$, CD on $(0, 1.124)$, and CU on $\left(1.124, \frac{\pi}{2}\right)$. Since f is periodic, this behavior repeats on every interval of length π. f has IPs at $(-1.124 + n\pi, 0.890)$, $\left(n\pi, \frac{1}{2}\right)$, and $(1.124 + n\pi, 0.110)$.

23. (a) $f(x) = x^{1/x}$

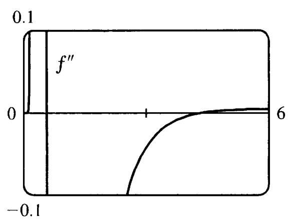


(b) Recall that $a^b = e^{b \ln a}$. $\lim\limits_{x \to 0^+} x^{1/x} = \lim\limits_{x \to 0^+} e^{(1/x)\ln x}$. As $x \to 0^+$, $\dfrac{\ln x}{x} \to -\infty$, so $x^{1/x} = e^{(1/x)\ln x} \to 0$. This indicates that there is a hole at $(0, 0)$. As $x \to \infty$, we have the indeterminate form ∞^0. $\lim\limits_{x \to \infty} x^{1/x} = \lim\limits_{x \to \infty} e^{(1/x)\ln x}$, but $\lim\limits_{x \to \infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x \to \infty} \dfrac{1/x}{1} = 0$, so $\lim\limits_{x \to \infty} x^{1/x} = e^0 = 1$. This indicates that $y = 1$ is a HA.

(c) Estimated maximum: $(2.72, 1.45)$. No estimated minimum. We use logarithmic differentiation to find any critical numbers. $y = x^{1/x} \;\Rightarrow\; \ln y = \dfrac{1}{x} \ln x \;\Rightarrow\; \dfrac{y'}{y} = \dfrac{1}{x} \cdot \dfrac{1}{x} + (\ln x)\left(-\dfrac{1}{x^2}\right) \;\Rightarrow$

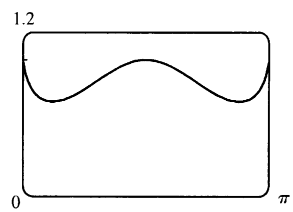
$y' = x^{1/x}\left(\dfrac{1 - \ln x}{x^2}\right) = 0 \;\Rightarrow\; \ln x = 1 \;\Rightarrow\; x = e$. For $0 < x < e$, $y' > 0$ and for $x > e$, $y' < 0$, so $f(e) = e^{1/e}$ is a local maximum value. This point is approximately $(2.7183, 1.4447)$, which agrees with our estimate.

(d)



From the graph, we see that $f''(x) = 0$ at $x \approx 0.58$ and $x \approx 4.37$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

24. (a) $f(x) = (\sin x)^{\sin x}$ is continuous where $\sin x > 0$, that is, on intervals of the form $(2n\pi, (2n+1)\pi)$, so we have graphed f on $(0, \pi)$.



(b) $y = (\sin x)^{\sin x} \;\Rightarrow\; \ln y = \sin x \ln \sin x$, so

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \sin x \ln \sin x = \lim_{x \to 0^+} \frac{\ln \sin x}{\csc x} \overset{\text{H}}{=} \lim_{x \to 0^+} \frac{\cot x}{-\csc x \cot x} = \lim_{x \to 0^+} (-\sin x) = 0$$

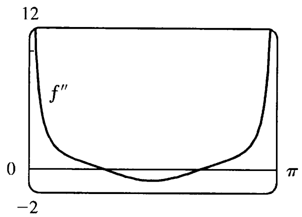
$\Rightarrow \quad \lim\limits_{x \to 0^+} y = e^0 = 1$.

(c) It appears that we have a local maximum at $(1.57, 1)$ and local minima at $(0.38, 0.69)$ and $(2.76, 0.69)$. $y = (\sin x)^{\sin x} \;\Rightarrow\; \ln y = \sin x \ln \sin x \;\Rightarrow$

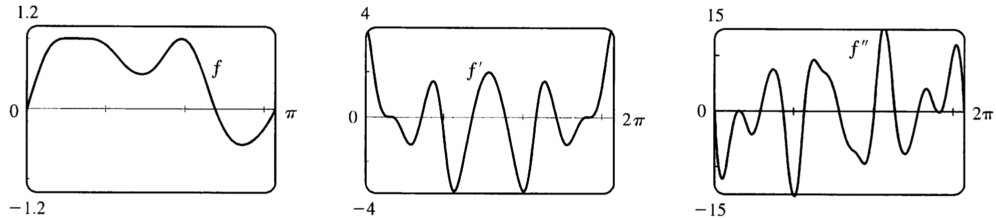
$\dfrac{y'}{y} = (\sin x)\left(\dfrac{\cos x}{\sin x}\right) + (\ln \sin x)\cos x = \cos x(1 + \ln \sin x) \;\Rightarrow\; y' = (\sin x)^{\sin x}(\cos x)(1 + \ln \sin x)$.

$y' = 0 \;\Rightarrow\; \cos x = 0$ or $\ln \sin x = -1 \;\Rightarrow\; x_2 = \frac{\pi}{2}$ or $\sin x = e^{-1}$. On $(0, \pi)$, $\sin x = e^{-1} \;\Rightarrow$ $x_1 = \sin^{-1}(e^{-1})$ and $x_3 = \pi - \sin^{-1}(e^{-1})$. Approximating these points gives us

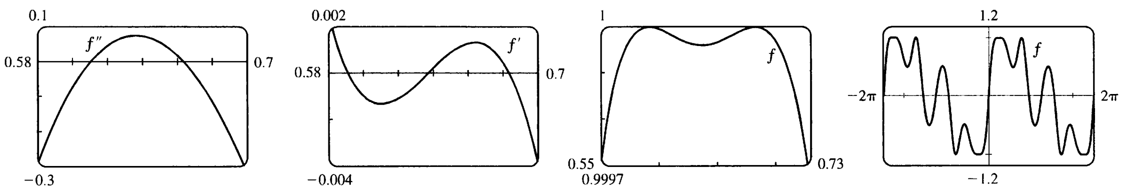
$(x_1, f(x_1)) \approx (0.3767, 0.6922)$, $(x_2, f(x_2)) \approx (1.5708, 1)$, and $(x_3, f(x_3)) \approx (2.7649, 0.6922)$. The approximations confirm our estimates.

(d) 

From the graph, we see that $f''(x) = 0$ at $x \approx 0.94$ and $x \approx 2.20$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

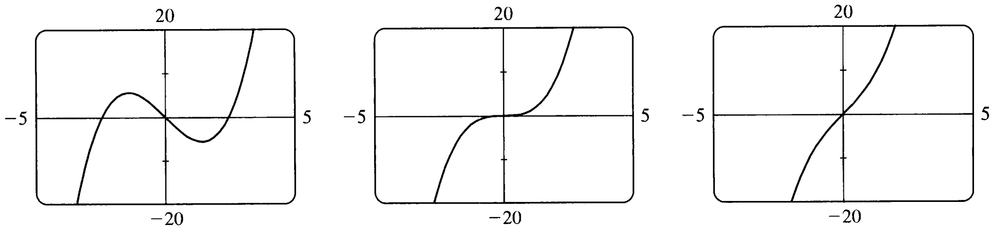
25. 

From the graph of $f(x) = \sin(x + \sin 3x)$ in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$, it looks like f has two maxima and two minima. If we calculate and graph $f'(x) = [\cos(x + \sin 3x)]\,(1 + 3\cos 3x)$ on $[0, 2\pi]$, we see that the graph of f' appears to be almost tangent to the x-axis at about $x = 0.7$. The graph of $f'' = -\,[\sin(x + \sin 3x)]\,(1 + 3\cos 3x)^2 + \cos(x + \sin 3x)(-9\sin 3x)$ is even more interesting near this x-value: it seems to just touch the x-axis.



If we zoom in on this place on the graph of f'', we see that f'' actually does cross the axis twice near $x = 0.65$, indicating a change in concavity for a very short interval. If we look at the graph of f' on the same interval, we see that it changes sign three times near $x = 0.65$, indicating that what we had thought was a broad extremum at about $x = 0.7$ actually consists of three extrema (two maxima and a minimum). These maximum values are roughly $f(0.59) = 1$ and $f(0.68) = 1$, and the minimum value is roughly $f(0.64) = 0.99996$. There are also a maximum value of about $f(1.96) = 1$ and minimum values of about $f(1.46) = 0.49$ and $f(2.73) = -0.51$. The points of inflection on $(0, \pi)$ are about $(0.61, 0.99998)$, $(0.66, 0.99998)$, $(1.17, 0.72)$, $(1.75, 0.77)$, and $(2.28, 0.34)$. On $(\pi, 2\pi)$, they are about $(4.01, -0.34)$, $(4.54, -0.77)$, $(5.11, -0.72)$, $(5.62, -0.99998)$, and $(5.67, -0.99998)$. There are also IP at $(0, 0)$ and $(\pi, 0)$. Note that the function is odd and periodic with period 2π, and it is also rotationally symmetric about all points of the form $((2n + 1)\pi, 0)$, n an integer.

26. $f(x) = x^3 + cx = x(x^2 + c) \quad \Rightarrow \quad f'(x) = 3x^2 + c \quad \Rightarrow \quad f''(x) = 6x$



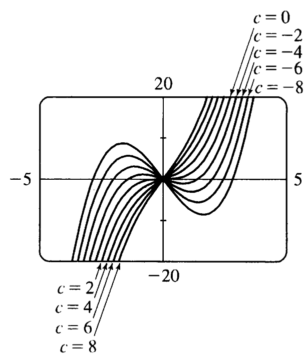
$c = -6$ $\qquad$ $c = 0$ $\qquad$ $c = 6$

x-intercepts: When $c \geq 0$, 0 is the only x-intercept. When $c < 0$, the x-intercepts are 0 and $\pm\sqrt{-c}$.
y-intercept $= f(0) = 0$. f is odd, so the graph is symmetric with respect to the origin. $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. The origin is the only inflection point.

If $c > 0$, then $f'(x) > 0$ for all x, so f is increasing and has no local maximum or minimum.

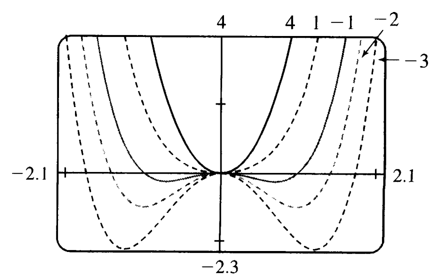
If $c = 0$, then $f'(x) \geq 0$ with equality at $x = 0$, so again f is increasing and has no local maximum or minimum.

If $c < 0$, then $f'(x) = 3[x^2 - (-c/3)] = 3\left(x + \sqrt{-c/3}\right)\left(x - \sqrt{-c/3}\right)$, so $f'(x) > 0$ on $\left(-\infty, -\sqrt{-c/3}\right)$ and $\left(\sqrt{-c/3}, \infty\right)$; $f'(x) < 0$ on $\left(-\sqrt{-c/3}, \sqrt{-c/3}\right)$. It follows that $f\left(-\sqrt{-c/3}\right) = -\frac{2}{3}c\sqrt{-c/3}$ is a local maximum value and $f\left(\sqrt{-c/3}\right) = \frac{2}{3}c\sqrt{-c/3}$ is a local minimum value. As c decreases (toward more negative values), the local maximum and minimum move further apart.



There is no absolute maximum or minimum value. The only transitional value of c corresponding to a change in character of the graph is $c = 0$.

27. $f(x) = x^4 + cx^2 = x^2(x^2 + c)$. Note that f is an even function. For $c \geq 0$, the only x-intercept is the point $(0, 0)$. We calculate $f'(x) = 4x^3 + 2cx = 4x(x^2 + \frac{1}{2}c) \quad \Rightarrow \quad f''(x) = 12x^2 + 2c$. If $c \geq 0$, $x = 0$ is the only critical point and there is no inflection point. As we can see from the examples, there is no change in the basic shape of the graph for $c \geq 0$; it merely becomes steeper as c increases. For $c = 0$, the graph is the simple curve $y = x^4$. For $c < 0$, there are x-intercepts at 0 and at $\pm\sqrt{-c}$. Also, there is a maximum at $(0, 0)$, and there are minima at $\left(\pm\sqrt{-\frac{1}{2}c}, -\frac{1}{4}c^2\right)$. As $c \to -\infty$, the x-coordinates of these minima get larger in absolute value, and the minimum points move downward. There are inflection points at $\left(\pm\sqrt{-\frac{1}{6}c}, -\frac{5}{36}c^2\right)$, which also move away from the origin as $c \to -\infty$.



28. We need only consider the function $f(x) = x^2\sqrt{c^2 - x^2}$ for $c \geq 0$, because if c is replaced by $-c$, the function is unchanged. For $c = 0$, the graph consists of the single point $(0, 0)$. The domain of f is $[-c, c]$, and the graph of f is symmetric about the y-axis.

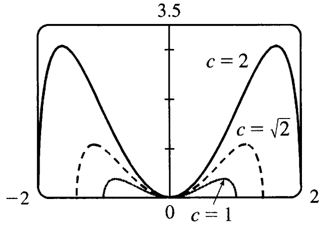
$$f'(x) = 2x\sqrt{c^2 - x^2} + x^2 \frac{-2x}{2\sqrt{c^2 - x^2}} = 2x\sqrt{c^2 - x^2} - \frac{x^3}{\sqrt{c^2 - x^2}} = \frac{2x(c^2 - x^2) - x^3}{\sqrt{c^2 - x^2}} = -\frac{3x\left(x^2 - \frac{2}{3}c^2\right)}{\sqrt{c^2 - x^2}}.$$

So we see that all members of the family of curves have horizontal tangents at $x = 0$, since $f'(0) = 0$ for all $c > 0$. Also, the tangents to all the curves become very steep as $x \to \pm c$, since $\lim\limits_{x \to -c^+} f'(x) = \infty$ and $\lim\limits_{x \to c^-} f'(x) = -\infty$. We set $f'(x) = 0 \iff x = 0$ or $x^2 - \frac{2}{3}c^2 = 0$, so the absolute maximum values are $f\left(\pm\sqrt{\frac{2}{3}}c\right) = \frac{2}{3\sqrt{3}}c^3$.

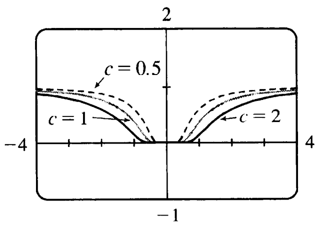
$$f''(x) = \frac{(-9x^2 + 2c^2)\sqrt{c^2 - x^2} - (-3x^3 + 2c^2x)\left(-x/\sqrt{c^2 - x^2}\right)}{c^2 - x^2} = \frac{6x^4 - 9c^2x^2 + 2c^4}{(c^2 - x^2)^{3/2}}.$$

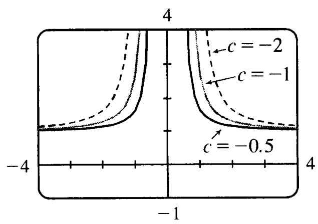
Using the quadratic formula, we find that $f''(x) = 0 \iff x^2 = \dfrac{9c^2 \pm c^2\sqrt{33}}{12}$. Since $-c < x < c$, we take $x^2 = \frac{9 - \sqrt{33}}{12}c^2$, so the inflection points are $\left(\pm\sqrt{\frac{9 - \sqrt{33}}{12}}c, \frac{(9 - \sqrt{33})(\sqrt{33} - 3)}{144}c^3\right)$.

From these calculations we can see that the maxima and the points of inflection get both horizontally and vertically further from the origin as c increases. Since all of the functions have two maxima and two inflection points, we see that the basic shape of the curve does not change as c changes.



29.





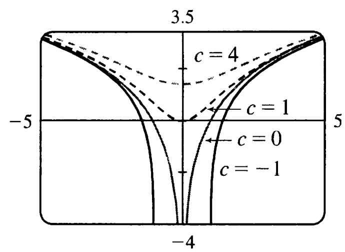
$c = 0$ is a transitional value—we get the graph of $y = 1$. For $c > 0$, we see that there is a HA at $y = 1$, and that the graph spreads out as c increases. At first glance there appears to be a minimum at $(0, 0)$, but $f(0)$ is undefined, so there is no minimum or maximum. For $c < 0$, we still have the HA at $y = 1$, but the range is $(1, \infty)$ rather than $(0, 1)$. We also have a VA at $x = 0$. $f(x) = e^{-c/x^2} \Rightarrow f'(x) = e^{-c/x^2}(-2c/x^3) \Rightarrow$

$f''(x) = \dfrac{2c(2c - 3x^2)}{x^6 e^{c/x^2}}$. $f'(x) \neq 0$ and $f'(x)$ exists for all $x \neq 0$ (and 0 is not in the domain of f), so there are no maxima or minima. $f''(x) = 0 \Rightarrow x = \pm\sqrt{2c/3}$, so if $c > 0$, the inflection points spread out as c increases, and if $c < 0$, there are no IP. For $c > 0$, there are IP at $\left(\pm\sqrt{2c/3}, e^{-3/2}\right)$. Note that the y-coordinate of the IP is constant.

30. We see that if $c \le 0$, $f(x) = \ln(x^2 + c)$ is only defined for $x^2 > -c \;\Rightarrow\; |x| > \sqrt{-c}$, and

$\lim\limits_{x\to\sqrt{-c}^+} f(x) = \lim\limits_{x\to-\sqrt{-c}^-} f(x) = -\infty$, since $\ln y \to -\infty$ as $y \to 0$. Thus, for $c < 0$, there are vertical asymptotes at $x = \pm\sqrt{c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c = 0$, $\lim\limits_{x\to 0} f(x) = -\infty$, so there is a vertical asymptote at $x = 0$. If $c > 0$, there are no asymptotes. To find the extrema and inflection points, we differentiate: $f(x) = \ln(x^2 + c) \;\Rightarrow\; f'(x) = \dfrac{1}{x^2 + c}(2x)$, so by the First Derivative Test there is a local and absolute minimum at $x = 0$. Differentiating again, we get

$$f''(x) = \frac{1}{x^2 + c}(2) + 2x\left[-\left(x^2 + c\right)^{-2}(2x)\right] = \frac{2(c - x^2)}{(x^2 + c)^2}.$$

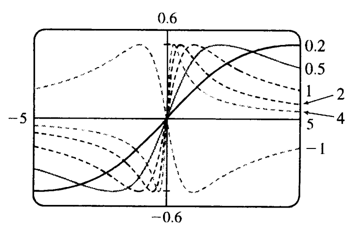


Now if $c \le 0$, f'' is always negative, so f is concave down on both of the intervals on which it is defined. If $c > 0$, then f'' changes sign when $c = x^2 \;\Leftrightarrow x = \pm\sqrt{c}$. So for $c > 0$ there are inflection points at $x = \pm\sqrt{c}$, and as c increases, the inflection points get further apart.

31. Note that $c = 0$ is a transitional value at which the graph consists of the x-axis. Also, we can see that if we substitute $-c$ for c, the function $f(x) = \dfrac{cx}{1 + c^2x^2}$ will be reflected in the x-axis, so we investigate only positive values of c (except $c = -1$, as a demonstration of this reflective property). Also, f is an odd function. $\lim\limits_{x\to\pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote for all c. We calculate

$f'(x) = \dfrac{(1 + c^2x^2)c - cx(2c^2x)}{(1 + c^2x^2)^2} = -\dfrac{c(c^2x^2 - 1)}{(1 + c^2x^2)^2}$. $f'(x) = 0 \;\Leftrightarrow\; c^2x^2 - 1 = 0 \;\Leftrightarrow\; x = \pm 1/c$. So there is an absolute maximum value of $f(1/c) = \frac{1}{2}$ and an absolute minimum value of $f(-1/c) = -\frac{1}{2}$. These extrema have the same value regardless of c, but the maximum points move closer to the y-axis as c increases.

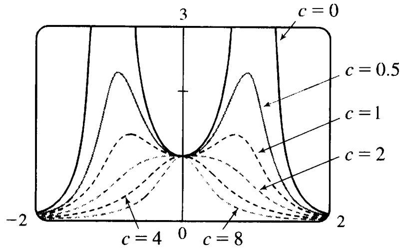
$$\begin{aligned} f''(x) &= \frac{(-2c^3x)(1 + c^2x^2)^2 - (-c^3x^2 + c)[2(1 + c^2x^2)(2c^2x)]}{(1 + c^2x^2)^4} \\ &= \frac{(-2c^3x)(1 + c^2x^2) + (c^3x^2 - c)(4c^2x)}{(1 + c^2x^2)^3} = \frac{2c^3x(c^2x^2 - 3)}{(1 + c^2x^2)^3} \end{aligned}$$



$f''(x) = 0 \;\Leftrightarrow\; x = 0$ or $\pm\sqrt{3}/c$, so there are inflection points at $(0, 0)$ and at $\left(\pm\sqrt{3}/c, \pm\sqrt{3}/4\right)$.

Again, the y-coordinate of the inflection points does not depend on c, but as c increases, both inflection points approach the y-axis.

32. Note that $f(x) = \dfrac{1}{(1-x^2)^2 + cx^2}$ is an even function, and also that $\lim\limits_{x\to\pm\infty} f(x) = 0$ for any value of c, so $y = 0$ is a horizontal asymptote. We calculate the derivatives:



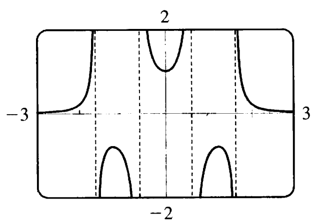
$$f'(x) = \frac{-4(1-x^2)x + 2cx}{\left[(1-x^2)^2 + cx^2\right]^2} = \frac{4x\left[x^2 + \left(\frac{1}{2}c - 1\right)\right]}{\left[(1-x^2)^2 + cx^2\right]^2}, \text{ and}$$

$$f''(x) = 2\frac{10x^6 + (9c-18)x^4 + (3c^2 - 12c + 6)x^2 + 2 - c}{\left[x^4 + (c-2)x^2 + 1\right]^3}.$$

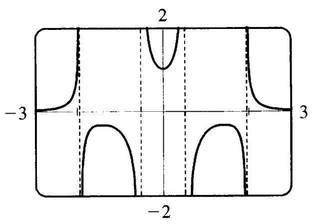
We first consider the case $c > 0$. Then the denominator of f' is positive, that is, $\left(1-x^2\right)^2 + cx^2 > 0$ for all x, so f has domain $\mathbb{R}$ and also $f > 0$. If $\frac{1}{2}c - 1 \geq 0$; that is, $c \geq 2$, then the only critical point is $f(0) = 1$, a maximum. Graphing a few examples for $c \geq 2$ shows that there are two IP which approach the y-axis as $c \to \infty$.

$c = 2$ and $c = 0$ are transitional values of c at which the shape of the curve changes. For $0 < c < 2$, there are three critical points: $f(0) = 1$, a minimum value, and $f\left(\pm\sqrt{1 - \frac{1}{2}c}\right) = \dfrac{1}{c(1-c/4)}$, both maximum values. As c decreases from 2 to 0, the maximum values get larger and larger, and the x-values at which they occur go from 0 to ± 1. Graphs show that there are four inflection points for $0 < c < 2$, and that they get farther away from the origin, both vertically and horizontally, as $c \to 0^+$. For $c = 0$, the function is simply asymptotic to the x-axis and to the lines $x = \pm 1$, approaching $+\infty$ from both sides of each. The y-intercept is 1, and $(0, 1)$ is a local minimum. There are no inflection points. Now if $c < 0$, we can write

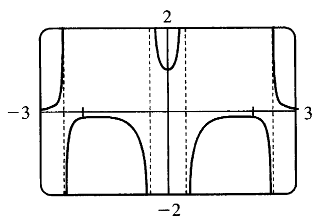
$$f(x) = \frac{1}{(1-x^2)^2 + cx^2} = \frac{1}{(1-x^2)^2 - \left(\sqrt{-c}\,x\right)^2} = \frac{1}{\left(x^2 - \sqrt{-c}\,x - 1\right)\left(x^2 + \sqrt{-c}\,x - 1\right)}.$$ So f has vertical asymptotes where $x^2 \pm \sqrt{-c}\,x - 1 = 0 \;\Leftrightarrow\; x = \left(-\sqrt{-c} \pm \sqrt{4-c}\right)/2$ or $x = \left(\sqrt{-c} \pm \sqrt{4-c}\right)/2$. As c decreases, the two exterior asymptotes move away from the origin, while the two interior ones move toward it. We graph a few examples to see the behavior of the graph near the asymptotes, and the nature of the critical points $x = 0$ and $x = \pm\sqrt{1 - \frac{1}{2}c}$:



$c = -1$



$c = -2$



$c = -4$

We see that there is one local minimum value, $f(0) = 1$, and there are two local maximum values, $f\left(\pm\sqrt{1 - \frac{1}{2}c}\right) = \dfrac{1}{c(1-c/4)}$ as before. As c decreases, the x-values at which these maxima occur get larger, and the maximum values themselves approach 0, though they are always negative.

33. $f(x) = cx + \sin x \quad \Rightarrow \quad f'(x) = c + \cos x \quad \Rightarrow \quad f''(x) = -\sin x$

$f(-x) = -f(x)$, so f is an odd function and its graph is symmetric with respect to the origin.

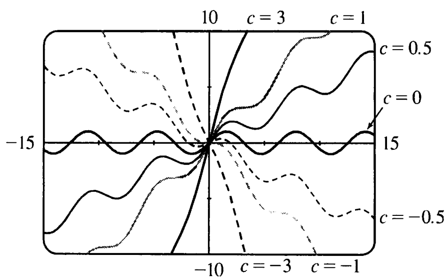
$f(x) = 0 \quad \Leftrightarrow \quad \sin x = -cx$, so 0 is always an x-intercept.

$f'(x) = 0 \quad \Leftrightarrow \quad \cos x = -c$, so there is no critical number when $|c| > 1$. If $|c| \le 1$, then there are infinitely many critical numbers. If x_1 is the unique solution of $\cos x = -c$ in the interval $[0, \pi]$, then the critical numbers are $2n\pi \pm x_1$, where n ranges over the integers. (Special cases: When $c = 1$, $x_1 = 0$; when $c = 0$, $x = \frac{\pi}{2}$; and when $c = -1$, $x_1 = \pi$.)

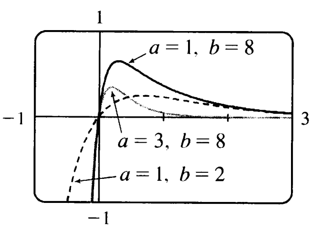
$f''(x) < 0 \quad \Leftrightarrow \quad \sin x > 0$, so f is CD on intervals of the form $(2n\pi, (2n+1)\pi)$. f is CU on intervals of the form $((2n-1)\pi, 2n\pi)$. The inflection points of f are the points $(2n\pi, 2n\pi c)$, where n is an integer.

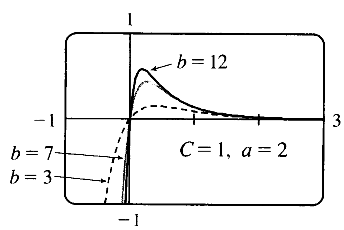
If $c \ge 1$, then $f'(x) \ge 0$ for all x, so f is increasing and has no extremum. If $c \le -1$, then $f'(x) \le 0$ for all x, so f is decreasing and has no extremum. If $|c| < 1$, then $f'(x) > 0 \quad \Leftrightarrow \quad \cos x > -c \quad \Leftrightarrow \quad x$ is in an interval of the form $(2n\pi - x_1, 2n\pi + x_1)$ for some integer n. These are the intervals on which f is increasing. Similarly, we find that f is decreasing on the intervals of the form $(2n\pi + x_1, 2(n+1)\pi - x_1)$. Thus, f has local maxima at the points $2n\pi + x_1$, where f has the values $c(2n\pi + x_1) + \sin x_1 = c(2n\pi + x_1) + \sqrt{1 - c^2}$, and f has local minima at the points $2n\pi - x_1$, where we have $f(2n\pi - x_1) = c(2n\pi - x_1) - \sin x_1 = c(2n\pi - x_1) - \sqrt{1 - c^2}$.

The transitional values of c are -1 and 1. The inflection points move vertically, but not horizontally, when c changes. When $|c| \ge 1$, there is no extremum. For $|c| < 1$, the maxima are spaced 2π apart horizontally, as are the minima. The horizontal spacing between maxima and adjacent minima is regular (and equals π) when $c = 0$, but the horizontal space between a local maximum and the nearest local minimum shrinks as $|c|$ approaches 1.



34. For $f(t) = C\left(e^{-at} - e^{-bt}\right)$, C affects only vertical stretching, so we let $C = 1$. From the first figure, we notice that the graphs all pass through the origin, approach the t-axis as t increases, and approach $-\infty$ as $t \to -\infty$. Next we let $a = 2$ and produce the second figure.





Here, as b increases, the slope of the tangent at the origin increases and the local maximum value increases.
$f(t) = e^{-2t} - e^{-bt} \quad \Rightarrow \quad f'(t) = be^{-bt} - 2e^{-2t}$. $f'(0) = b - 2$, which increases as b increases.

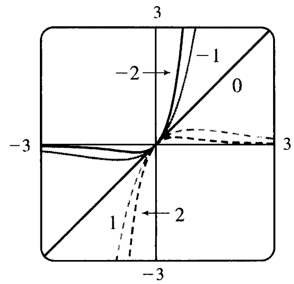
$f'(t) = 0 \quad \Rightarrow \quad be^{-bt} = 2e^{-2t} \quad \Rightarrow \quad \dfrac{b}{2} = e^{(b-2)t} \quad \Rightarrow \quad \ln \dfrac{b}{2} = (b-2)t \quad \Rightarrow \quad t = t_1 = \dfrac{\ln b - \ln 2}{b-2}$, which decreases as b increases (the maximum is getting closer to the y-axis). $f(t_1) = \dfrac{(b-2)2^{2/(b-2)}}{b^{1+2/(b-2)}}$. We can show that this value increases as b increases by considering it to be a function of b and graphing its derivative with respect to b, which is always positive.

35. If $c < 0$, then $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \dfrac{x}{e^{cx}} \overset{\text{H}}{=} \lim\limits_{x \to -\infty} \dfrac{1}{ce^{cx}} = 0$, and $\lim\limits_{x \to \infty} f(x) = \infty$.

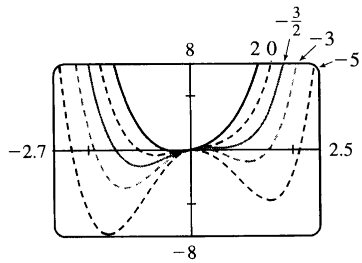
If $c > 0$, then $\lim\limits_{x \to -\infty} f(x) = -\infty$, and $\lim\limits_{x \to \infty} f(x) \overset{\text{H}}{=} \lim\limits_{x \to \infty} \dfrac{1}{ce^{cx}} = 0$.

If $c = 0$, then $f(x) = x$, so $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$ respectively.

So we see that $c = 0$ is a transitional value. We now exclude the case $c = 0$, since we know how the function behaves in that case. To find the maxima and minima of f, we differentiate: $f(x) = xe^{-cx} \quad \Rightarrow$ $f'(x) = x(-ce^{-cx}) + e^{-cx} = (1 - cx)e^{-cx}$. This is 0 when $1 - cx = 0 \quad \Leftrightarrow \quad x = 1/c$. If $c < 0$ then this represents a minimum value of $f(1/c) = 1/(ce)$, since $f'(x)$ changes from negative to positive at $x = 1/c$; and if $c > 0$, it represents a maximum value. As $|c|$ increases, the maximum or minimum point gets closer to the origin. To find the inflection points, we differentiate again: $f'(x) = e^{-cx}(1 - cx) \quad \Rightarrow$ $f''(x) = e^{-cx}(-c) + (1 - cx)(-ce^{-cx}) = (cx - 2)ce^{-cx}$. This changes sign when $cx - 2 = 0 \quad \Leftrightarrow \quad x = 2/c$. So as $|c|$ increases, the points of inflection get closer to the origin.

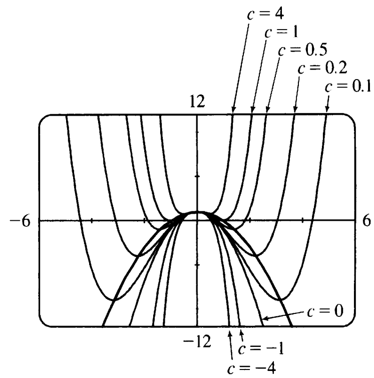


36. For $c = 0$, there is no inflection point; the curve is CU everywhere. If c increases, the curve simply becomes steeper, and there are still no inflection points. If c starts at 0 and decreases, a slight upward bulge appears near $x = 0$, so that there are two inflection points for any $c < 0$. This can be seen algebraically by calculating the second derivative: $f(x) = x^4 + cx^2 + x \quad \Rightarrow \quad f'(x) = 4x^3 + 2cx + 1 \quad \Rightarrow \quad f''(x) = 12x^2 + 2c$. Thus, $f''(x) > 0$ when $c > 0$. For $c < 0$, there are inflection points when $x = \pm\sqrt{-\frac{1}{6}c}$. For $c = 0$, the graph has one critical number, at the absolute minimum somewhere around $x = -0.6$. As c increases, the number of critical points does not change. If c instead decreases from 0, we see that the graph eventually sprouts another local minimum, to the right of the origin, somewhere between $x = 1$ and $x = 2$. Consequently, there is also a maximum near $x = 0$. After a bit of experimentation, we find that at $c = -1.5$, there appear to be two critical numbers: the absolute minimum at about $x = -1$, and a horizontal tangent with no extremum at about $x = 0.5$. For any c smaller than this there will be 3 critical points, as shown in the graphs with $c = -3$ and with $c = -5$. To prove this algebraically, we calculate $f'(x) = 4x^3 + 2cx + 1$. Now if we substitute our value of $c = -1.5$, the formula for $f'(x)$ becomes $4x^3 - 3x + 1 = (x + 1)(2x - 1)^2$. This has a double root at $x = \frac{1}{2}$, indicating that the function has two critical points: $x = -1$ and $x = \frac{1}{2}$, just as we had guessed from the graph.



37. (a) $f(x) = cx^4 - 2x^2 + 1$. For $c = 0$, $f(x) = -2x^2 + 1$, a parabola whose vertex, $(0, 1)$, is the absolute maximum. For $c > 0$, $f(x) = cx^4 - 2x^2 + 1$ opens upward with two minimum points. As $c \to 0$, the minimum points spread apart and move downward; they are below the x-axis for $0 < c < 1$ and above for $c > 1$. For $c < 0$, the graph opens downward, and has an absolute maximum at $x = 0$ and no local minimum.

(b) $f'(x) = 4cx^3 - 4x = 4cx(x^2 - 1/c)$ $(c \neq 0)$. If $c \leq 0$, 0 is the only critical number. $f''(x) = 12cx^2 - 4$, so $f''(0) = -4$ and there is a local maximum at $(0, f(0)) = (0, 1)$, which lies on $y = 1 - x^2$. If $c > 0$, the critical numbers are 0 and $\pm 1/\sqrt{c}$. As before, there is a local maximum at $(0, f(0)) = (0, 1)$, which lies on $y = 1 - x^2$. $f''(\pm 1/\sqrt{c}) = 12 - 4 = 8 > 0$, so there is a local minimum at $x = \pm 1/\sqrt{c}$. Here $f(\pm 1/\sqrt{c}) = c(1/c^2) - 2/c + 1 = -1/c + 1$. But $(\pm 1/\sqrt{c}, -1/c + 1)$ lies on $y = 1 - x^2$ since $1 - (\pm 1/\sqrt{c})^2 = 1 - 1/c$.

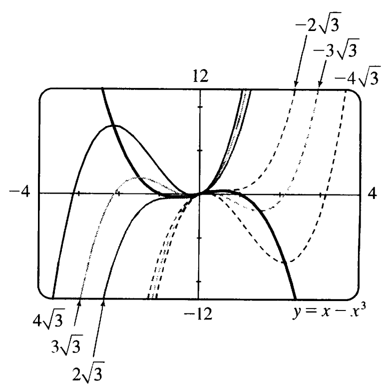


38. (a) $f(x) = 2x^3 + cx^2 + 2x \Rightarrow f'(x) = 6x^2 + 2cx + 2 = 2(3x^2 + cx + 1)$. $f'(x) = 0 \Leftrightarrow$ $x = \dfrac{-c \pm \sqrt{c^2 - 12}}{6}$. So f has critical points $\Leftrightarrow c^2 - 12 \geq 0 \Leftrightarrow |c| \geq 2\sqrt{3}$. For $c = \pm 2\sqrt{3}$, $f'(x) \geq 0$ on $(-\infty, \infty)$, so f' does not change signs at $-c/6$, and there is no extremum. If $c^2 - 12 > 0$, then f' changes from positive to negative at $x = \dfrac{-c - \sqrt{c^2 - 12}}{6}$ and from negative to positive at $x = \dfrac{-c + \sqrt{c^2 - 12}}{6}$. So f has a local maximum at $x = \dfrac{-c - \sqrt{c^2 - 12}}{6}$ and a local minimum at $x = \dfrac{-c + \sqrt{c^2 - 12}}{6}$.

(b) Let x_0 be a critical number for $f(x)$. Then $f'(x_0) = 0 \Rightarrow$ $3x_0^2 + cx_0 + 1 = 0 \Leftrightarrow c = \dfrac{-1 - 3x_0^2}{x_0}$. Now

$$f(x_0) = 2x_0^3 + cx_0^2 + 2x_0 = 2x_0^3 + x_0^2\left(\frac{-1 - 3x_0^2}{x_0}\right) + 2x_0$$
$$= 2x_0^3 - x_0 - 3x_0^3 + 2x_0 = x_0 - x_0^3$$

So the point is $(x_0, y_0) = (x_0, x_0 - x_0^3)$; that is, the point lies on the curve $y = x - x^3$.



4.7 Optimization Problems

1. (a)

First Number	Second Number	Product
1	22	22
2	21	42
3	20	60
4	19	76
5	18	90
6	17	102
7	16	112
8	15	120
9	14	126
10	13	130
11	12	132

We needn't consider pairs where the first number is larger than the second, since we can just interchange the numbers in such cases. The answer appears to be 11 and 12, but we have considered only integers in the table.

(b) Call the two numbers x and y. Then $x + y = 23$, so $y = 23 - x$. Call the product P. Then $P = xy = x(23 - x) = 23x - x^2$, so we wish to maximize the function $P(x) = 23x - x^2$. Since $P'(x) = 23 - 2x$, we see that $P'(x) = 0 \quad \Leftrightarrow \quad x = \frac{23}{2} = 11.5$. Thus, the maximum value of P is $P(11.5) = (11.5)^2 = 132.25$ and it occurs when $x = y = 11.5$.

Or: Note that $P''(x) = -2 < 0$ for all x, so P is everywhere concave downward and the local maximum at $x = 11.5$ must be an absolute maximum.

2. The two numbers are $x + 100$ and x. Minimize $f(x) = (x + 100)x = x^2 + 100x$. $f'(x) = 2x + 100 = 0 \quad \Rightarrow$ $x = -50$. Since $f''(x) = 2 > 0$, there is an absolute minimum at $x = -50$. The two numbers are 50 and -50.

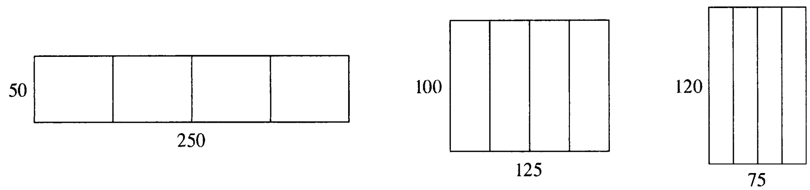
3. The two numbers are x and $\dfrac{100}{x}$, where $x > 0$. Minimize $f(x) = x + \dfrac{100}{x}$. $f'(x) = 1 - \dfrac{100}{x^2} = \dfrac{x^2 - 100}{x^2}$.
The critical number is $x = 10$. Since $f'(x) < 0$ for $0 < x < 10$ and $f'(x) > 0$ for $x > 10$, there is an absolute minimum at $x = 10$. The numbers are 10 and 10.

4. Let $x > 0$ and let $f(x) = x + 1/x$. We wish to minimize $f(x)$. Now
$f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{1}{x^2}(x^2 - 1) = \dfrac{1}{x^2}(x + 1)(x - 1)$, so the only critical number in $(0, \infty)$ is 1.
$f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$, so f has an absolute minimum at $x = 1$, and $f(1) = 2$.
Or: $f''(x) = 2/x^3 > 0$ for all $x > 0$, so f is concave upward everywhere and the critical point $(1, 2)$ must correspond to a local minimum for f.

5. If the rectangle has dimensions x and y, then its perimeter is $2x + 2y = 100$ m, so $y = 50 - x$. Thus, the area is $A = xy = x(50 - x)$. We wish to maximize the function $A(x) = x(50 - x) = 50x - x^2$, where $0 < x < 50$. Since $A'(x) = 50 - 2x = -2(x - 25)$, $A'(x) > 0$ for $0 < x < 25$ and $A'(x) < 0$ for $25 < x < 50$. Thus, A has an absolute maximum at $x = 25$, and $A(25) = 25^2 = 625\text{ m}^2$. The dimensions of the rectangle that maximize its area are $x = y = 25$ m. (The rectangle is a square.)

6. If the rectangle has dimensions x and y, then its area is $xy = 1000 \text{ m}^2$, so $y = 1000/x$. The perimeter $P = 2x + 2y = 2x + 2000/x$. We wish to minimize the function $P(x) = 2x + 2000/x$ for $x > 0$. $P'(x) = 2 - 2000/x^2 = (2/x^2)(x^2 - 1000)$, so the only critical number in the domain of P is $x = \sqrt{1000}$. $P''(x) = 4000/x^3 > 0$, so P is concave upward throughout its domain and $P(\sqrt{1000}) = 4\sqrt{1000}$ is an absolute minimum value. The dimensions of the rectangle with minimal perimeter are $x = y = \sqrt{1000} = 10\sqrt{10}$ m. (The rectangle is a square.)

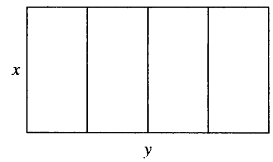
7. (a)



The areas of the three figures are 12,500, 12,500, and 9000 ft^2. There appears to be a maximum area of at least 12,500 ft^2.

(b) Let x denote the length of each of two sides and three dividers.

Let y denote the length of the other two sides.



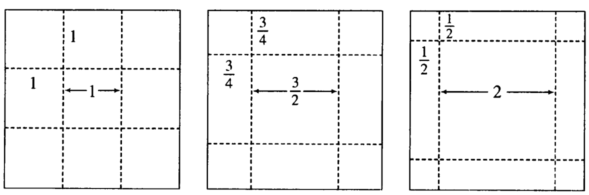
(c) Area $A = \text{length} \times \text{width} = y \cdot x$

(d) Length of fencing $= 750 \quad \Rightarrow \quad 5x + 2y = 750$

(e) $5x + 2y = 750 \quad \Rightarrow \quad y = 375 - \frac{5}{2}x \quad \Rightarrow \quad A(x) = \left(375 - \frac{5}{2}x\right)x = 375x - \frac{5}{2}x^2$

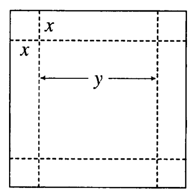
(f) $A'(x) = 375 - 5x = 0 \quad \Rightarrow \quad x = 75$. Since $A''(x) = -5 < 0$ there is an absolute maximum when $x = 75$. Then $y = \frac{375}{2} = 187.5$. The largest area is $75\left(\frac{375}{2}\right) = 14{,}062.5 \text{ ft}^2$. These values of x and y are between the values in the first and second figures in part (a). Our original estimate was low.

8. (a)



The volumes of the resulting boxes are 1, 1.6875, and 2 ft^3. There appears to be a maximum volume of at least 2 ft^3.

(b) Let x denote the length of the side of the square being cut out. Let y denote the length of the base.



(c) Volume $V = \text{length} \times \text{width} \times \text{height} \quad \Rightarrow \quad V = y \cdot y \cdot x = xy^2$

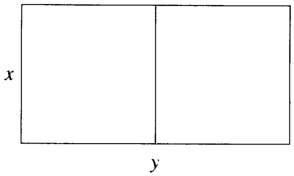
(d) Length of cardboard $= 3 \quad \Rightarrow \quad x + y + x = 3 \quad \Rightarrow \quad y + 2x = 3$

(e) $y + 2x = 3 \quad \Rightarrow \quad y = 3 - 2x \quad \Rightarrow \quad V(x) = x(3 - 2x)^2$

(f) $V(x) = x(3 - 2x)^2 \quad \Rightarrow$

$V'(x) = x \cdot 2(3 - 2x)(-2) + (3 - 2x)^2 \cdot 1 = (3 - 2x)[-4x + (3 - 2x)] = (3 - 2x)(-6x + 3)$, so the critical numbers are $x = \frac{3}{2}$ and $x = \frac{1}{2}$. Now $0 \leq x \leq \frac{3}{2}$ and $V(0) = V\left(\frac{3}{2}\right) = 0$, so the maximum is $V\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)(2)^2 = 2 \text{ ft}^3$, which is the value found from our third figure in part (a).

9.



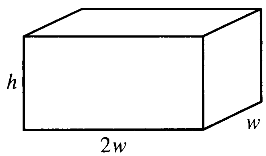
$xy = 1.5 \times 10^6$, so $y = 1.5 \times 10^6/x$. Minimize the amount of fencing, which is $3x + 2y = 3x + 2(1.5 \times 10^6/x) = 3x + 3 \times 10^6/x = F(x)$. $F'(x) = 3 - 3 \times 10^6/x^2 = 3(x^2 - 10^6)/x^2$. The critical number is $x = 10^3$ and $F'(x) < 0$ for $0 < x < 10^3$ and $F'(x) > 0$ if $x > 10^3$, so the absolute minimum occurs when $x = 10^3$ and $y = 1.5 \times 10^3$.

The field should be 1000 feet by 1500 feet with the middle fence parallel to the short side of the field.

10. Let b be the length of the base of the box and h the height. The volume is $32{,}000 = b^2 h \quad \Rightarrow \quad h = 32{,}000/b^2$. The surface area of the open box is $S = b^2 + 4hb = b^2 + 4(32{,}000/b^2)b = b^2 + 4(32{,}000)/b$. So $S'(b) = 2b - 4(32{,}000)/b^2 = 2(b^3 - 64{,}000)/b^2 = 0 \quad \Leftrightarrow \quad b = \sqrt[3]{64{,}000} = 40$. This gives an absolute minimum since $S'(b) < 0$ if $0 < b < 40$ and $S'(b) > 0$ if $b > 40$. The box should be $40 \times 40 \times 20$.

11. Let b be the length of the base of the box and h the height. The surface area is $1200 = b^2 + 4hb \quad \Rightarrow$ $h = (1200 - b^2)/(4b)$. The volume is $V = b^2 h = b^2(1200 - b^2)/4b = 300b - b^3/4 \quad \Rightarrow \quad V'(b) = 300 - \frac{3}{4}b^2$. $V'(b) = 0 \quad \Rightarrow \quad 300 = \frac{3}{4}b^2 \quad \Rightarrow \quad b^2 = 400 \quad \Rightarrow \quad b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Extreme Values (see page 334). If $b = 20$, then $h = (1200 - 20^2)/(4 \cdot 20) = 10$, so the largest possible volume is $b^2 h = (20)^2(10) = 4000\ \text{cm}^3$.

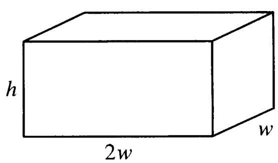
12.



$V = lwh \quad \Rightarrow \quad 10 = (2w)(w)h = 2w^2 h$, so $h = 5/w^2$. The cost is $10(2w^2) + 6[2(2wh) + 2(hw)] = 20w^2 + 36wh$, so $C(w) = 20w^2 + 36w(5/w^2) = 20w^2 + 180/w$.

$C'(w) = 40w - 180/w^2 = 40\left(w^3 - \frac{9}{2}\right)/w^2 \quad \Rightarrow \quad w = \sqrt[3]{\frac{9}{2}}$ is the critical number. There is an absolute minimum for C when $w = \sqrt[3]{\frac{9}{2}}$ since $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{9}{2}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{9}{2}}$. $C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 + \frac{180}{\sqrt[3]{9/2}} \approx \$163.54$.

13.



$10 = (2w)(w)h = 2w^2 h$, so $h = 5/w^2$. The cost is

$$C(w) = 10(2w^2) + 6[2(2wh) + 2hw] + 6(2w^2)$$
$$= 32w^2 + 36wh = 32w^2 + 180/w$$

$C'(w) = 64w - 180/w^2 = 4\left(16w^3 - 45\right)/w^2 \quad \Rightarrow \quad w = \sqrt[3]{\frac{45}{16}}$ is the critical number. $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{45}{16}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{45}{16}}$. The minimum cost is $C\left(\sqrt[3]{\frac{45}{16}}\right) = 32(2.8125)^{2/3} + 180/\sqrt{2.8125} \approx \$191.28$.

14. (a) Let the rectangle have sides x and y and area A, so $A = xy$ or $y = A/x$. The problem is to minimize the perimeter $= 2x + 2y = 2x + 2A/x = P(x)$. Now $P'(x) = 2 - 2A/x^2 = 2(x^2 - A)/x^2$. So the critical number is $x = \sqrt{A}$. Since $P'(x) < 0$ for $0 < x < \sqrt{A}$ and $P'(x) > 0$ for $x > \sqrt{A}$, there is an absolute minimum at $x = \sqrt{A}$. The sides of the rectangle are $\sqrt{A}$ and $A/\sqrt{A} = \sqrt{A}$, so the rectangle is a square.

(b) Let p be the perimeter and x and y the lengths of the sides, so $p = 2x + 2y \quad \Rightarrow \quad 2y = p - 2x \quad \Rightarrow$ $y = \frac{1}{2}p - x$. The area is $A(x) = x\left(\frac{1}{2}p - x\right) = \frac{1}{2}px - x^2$. Now $A'(x) = 0 \quad \Rightarrow \quad \frac{1}{2}p - 2x = 0 \quad \Rightarrow$

$2x = \frac{1}{2}p \;\Rightarrow\; x = \frac{1}{4}p$. Since $A''(x) = -2 < 0$, there is an absolute maximum for A when $x = \frac{1}{4}p$ by the Second Derivative Test. The sides of the rectangle are $\frac{1}{4}p$ and $\frac{1}{2}p - \frac{1}{4}p = \frac{1}{4}p$, so the rectangle is a square.

15. The distance from a point (x, y) on the line $y = 4x + 7$ to the origin is $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$. However, it is easier to work with the *square* of the distance; that is,

$D(x) = \left(\sqrt{x^2+y^2}\right)^2 = x^2 + y^2 = x^2 + (4x+7)^2$. Because the distance is positive, its minimum value will occur at the same point as the minimum value of D.

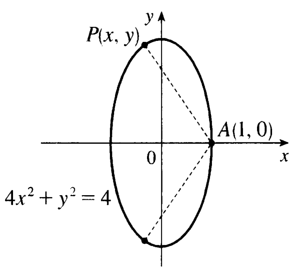
$D'(x) = 2x + 2(4x+7)(4) = 34x + 56$, so $D'(x) = 0 \;\Leftrightarrow\; x = -\frac{28}{17}$.

$D''(x) = 34 > 0$, so D is concave upward for all x. Thus, D has an absolute minimum at $x = -\frac{28}{17}$. The point closest to the origin is $(x, y) = \left(-\frac{28}{17}, 4\left(-\frac{28}{17}\right) + 7\right) = \left(-\frac{28}{17}, \frac{7}{17}\right)$.

16. The square of the distance from a point (x, y) on the line $y = -6x + 9$ to the point $(-3, 1)$ is

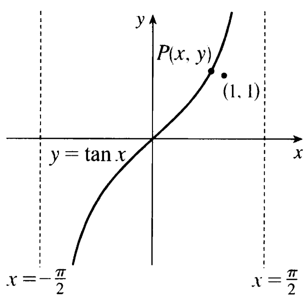
$D(x) = (x+3)^2 + (y-1)^2 = (x+3)^2 + (-6x+8)^2 = 37x^2 - 90x + 73$. $D'(x) = 74x - 90$, so $D'(x) = 0$ $\Leftrightarrow\; x = \frac{45}{37}$. $D''(x) = 74 > 0$, so D is concave upward for all x. Thus, D has an absolute minimum at $x = \frac{45}{37}$. The point on the line closest to $(-3, 1)$ is $\left(\frac{45}{37}, \frac{63}{37}\right)$.

17.



From the figure, we see that there are two points that are farthest away from $A(1, 0)$. The distance d from A to an arbitrary point $P(x, y)$ on the ellipse is $d = \sqrt{(x-1)^2 + (y-0)^2}$ and the square of the distance is $S = d^2 = x^2 - 2x + 1 + y^2 = x^2 - 2x + 1 + (4 - 4x^2) = -3x^2 - 2x + 5$. $S' = -6x - 2$ and $S' = 0 \;\Rightarrow\; x = -\frac{1}{3}$. Now $S'' = -6 < 0$, so we know that S has a maximum at $x = -\frac{1}{3}$. Since $-1 \le x \le 1$, $S(-1) = 4$, $S\left(-\frac{1}{3}\right) = \frac{16}{3}$, and $S(1) = 0$, we see that the maximum distance is $\sqrt{\frac{16}{3}}$. The corresponding y-values are $y = \pm\sqrt{4 - 4\left(-\frac{1}{3}\right)^2} = \pm\sqrt{\frac{32}{9}} = \pm\frac{4}{3}\sqrt{2} \approx \pm 1.89$. The points are $\left(-\frac{1}{3}, \pm\frac{4}{3}\sqrt{2}\right)$.

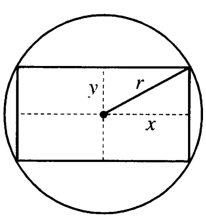
18.



The distance d from $(1, 1)$ to an arbitrary point $P(x, y)$ on the curve $y = \tan x$ is $d = \sqrt{(x-1)^2 + (y-1)^2}$ and the square of the distance is $S = d^2 = (x-1)^2 + (\tan x - 1)^2$.

$S' = 2(x-1) + 2(\tan x - 1)\sec^2 x$. Graphing S' on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ gives us a zero at $x \approx 0.82$, and so $\tan x \approx 1.08$. The point on $y = \tan x$ that is closest to $(1, 1)$ is approximately $(0.82, 1.08)$.

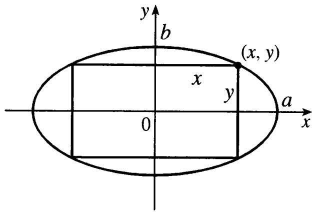
19.



The area of the rectangle is $(2x)(2y) = 4xy$. Also $r^2 = x^2 + y^2$ so $y = \sqrt{r^2 - x^2}$, so the area is $A(x) = 4x\sqrt{r^2 - x^2}$. Now

$A'(x) = 4\left(\sqrt{r^2 - x^2} - \dfrac{x^2}{\sqrt{r^2 - x^2}}\right) = 4\dfrac{r^2 - 2x^2}{\sqrt{r^2 - x^2}}$. The critical number is $x = \frac{1}{\sqrt{2}}r$. Clearly this gives a maximum.

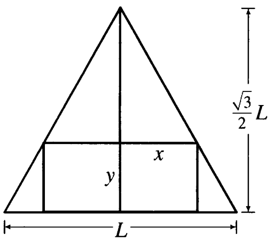
$y = \sqrt{r^2 - \left(\frac{1}{\sqrt{2}}r\right)^2} = \sqrt{\frac{1}{2}r^2} = \frac{1}{\sqrt{2}}r = x$, which tells us that the rectangle is a square. The dimensions are $2x = \sqrt{2}\,r$ and $2y = \sqrt{2}\,r$.

20. 

The area of the rectangle is $(2x)(2y) = 4xy$. Now $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ gives $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$, so we maximize $A(x) = 4\dfrac{b}{a}x\sqrt{a^2 - x^2}$.

$$A'(x) = \frac{4b}{a}\left[x \cdot \tfrac{1}{2}\left(a^2 - x^2\right)^{-1/2}(-2x) + \left(a^2 - x^2\right)^{1/2} \cdot 1\right]$$
$$= \frac{4b}{a}\left(a^2 - x^2\right)^{-1/2}\left[-x^2 + a^2 - x^2\right] = \frac{4b}{a\sqrt{a^2 - x^2}}\left[a^2 - 2x^2\right]$$

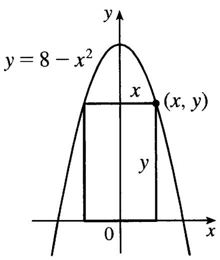
So the critical number is $x = \frac{1}{\sqrt{2}}a$, and this clearly gives a maximum. Then $y = \frac{1}{\sqrt{2}}b$, so the maximum area is $4\left(\frac{1}{\sqrt{2}}a\right)\left(\frac{1}{\sqrt{2}}b\right) = 2ab$.

21. 

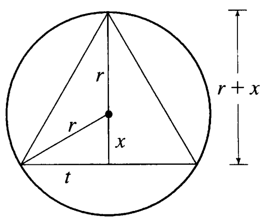
The height h of the equilateral triangle with sides of length L is $\frac{\sqrt{3}}{2}L$, since $h^2 + (L/2)^2 = L^2 \;\Rightarrow\; h^2 = L^2 - \frac{1}{4}L^2 = \frac{3}{4}L^2 \;\Rightarrow\; h = \frac{\sqrt{3}}{2}L$. Using similar triangles, $\dfrac{\frac{\sqrt{3}}{2}L - y}{x} = \dfrac{\frac{\sqrt{3}}{2}L}{L/2} = \sqrt{3} \;\Rightarrow$

$$\sqrt{3}\,x = \frac{\sqrt{3}}{2}L - y \;\Rightarrow\; y = \frac{\sqrt{3}}{2}L - \sqrt{3}\,x \;\Rightarrow\; y = \frac{\sqrt{3}}{2}(L - 2x).$$

The area of the inscribed rectangle is $A(x) = (2x)y = \sqrt{3}\,x(L - 2x) = \sqrt{3}\,Lx - 2\sqrt{3}\,x^2$, where $0 \le x \le L/2$. Now $0 = A'(x) = \sqrt{3}\,L - 4\sqrt{3}\,x \;\Rightarrow\; x = \sqrt{3}\,L/\left(4\sqrt{3}\right) = L/4$. Since $A(0) = A(L/2) = 0$, the maximum occurs when $x = L/4$, and $y = \frac{\sqrt{3}}{2}L - \frac{\sqrt{3}}{4}L = \frac{\sqrt{3}}{4}L$, so the dimensions are $L/2$ and $\frac{\sqrt{3}}{4}L$.

22. 

The rectangle has area $A(x) = 2xy = 2x\left(8 - x^2\right) = 16x - 2x^3$, where $0 \le x \le 2\sqrt{2}$. Now $A'(x) = 16 - 6x^2 = 0 \;\Rightarrow\; x = 2\sqrt{\frac{2}{3}}$. Since $A(0) = A\left(2\sqrt{2}\right) = 0$, there is a maximum when $x = 2\sqrt{\frac{2}{3}}$. Then $y = \frac{16}{3}$, so the rectangle has dimensions $4\sqrt{\frac{2}{3}}$ and $\frac{16}{3}$.

23. 

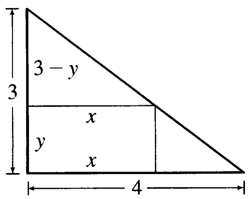
The area of the triangle is

$A(x) = \frac{1}{2}(2t)(r + x) = t(r + x) = \sqrt{r^2 - x^2}(r + x)$. Then

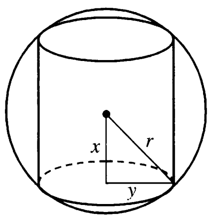
$$0 = A'(x) = r\frac{-2x}{2\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} + x\frac{-2x}{2\sqrt{r^2 - x^2}}$$
$$= -\frac{x^2 + rx}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} \;\Rightarrow$$

$$\frac{x^2 + rx}{\sqrt{r^2 - x^2}} = \sqrt{r^2 - x^2} \;\Rightarrow\; x^2 + rx = r^2 - x^2 \;\Rightarrow\; 0 = 2x^2 + rx - r^2 = (2x - r)(x + r) \;\Rightarrow$$

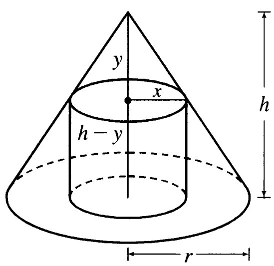
$x = \frac{1}{2}r$ or $x = -r$. Now $A(r) = 0 = A(-r) \Rightarrow$ the maximum occurs where $x = \frac{1}{2}r$, so the triangle has height $r + \frac{1}{2}r = \frac{3}{2}r$ and base $2\sqrt{r^2 - \left(\frac{1}{2}r\right)^2} = 2\sqrt{\frac{3}{4}r^2} = \sqrt{3}\,r$.

24. 

The rectangle has area xy. By similar triangles $\dfrac{3-y}{x} = \dfrac{3}{4} \Rightarrow$ $-4y + 12 = 3x$ or $y = -\frac{3}{4}x + 3$. So the area is $A(x) = x\left(-\frac{3}{4}x + 3\right) = -\frac{3}{4}x^2 + 3x$ where $0 \le x \le 4$. Now $0 = A'(x) = -\frac{3}{2}x + 3 \Rightarrow x = 2$ and $y = \frac{3}{2}$. Since $A(0) = A(4) = 0$, the maximum area is $A(2) = 2\left(\frac{3}{2}\right) = 3\text{ cm}^2$.

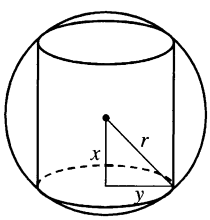
25. 

The cylinder has volume $V = \pi y^2(2x)$. Also $x^2 + y^2 = r^2 \Rightarrow$ $y^2 = r^2 - x^2$, so $V(x) = \pi\left(r^2 - x^2\right)(2x) = 2\pi\left(r^2x - x^3\right)$, where $0 \le x \le r$. $V'(x) = 2\pi\left(r^2 - 3x^2\right) = 0 \Rightarrow x = r/\sqrt{3}$. Now $V(0) = V(r) = 0$, so there is a maximum when $x = r/\sqrt{3}$ and $V\left(r/\sqrt{3}\right) = \pi\left(r^2 - r^2/3\right)\left(2r/\sqrt{3}\right) = 4\pi r^3/\left(3\sqrt{3}\right)$.

26. 

By similar triangles, $y/x = h/r$, so $y = hx/r$. The volume of the cylinder is $\pi x^2(h - y) = \pi hx^2 - (\pi h/r)x^3 = V(x)$. Now $V'(x) = 2\pi hx - (3\pi h/r)x^2 = \pi hx(2 - 3x/r)$. So $V'(x) = 0 \Rightarrow x = 0$ or $x = \frac{2}{3}r$. The maximum clearly occurs when $x = \frac{2}{3}r$ and then the volume is

$\pi hx^2 - (\pi h/r)x^3 = \pi hx^2(1 - x/r) = \pi\left(\frac{2}{3}r\right)^2 h\left(1 - \frac{2}{3}\right) = \frac{4}{27}\pi r^2 h$.

27. 

The cylinder has surface area

2(area of the base) + (lateral surface area)

$= 2\pi(\text{radius})^2 + 2\pi(\text{radius})(\text{height}) = 2\pi y^2 + 2\pi y(2x)$.

Now $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2 \Rightarrow y = \sqrt{r^2 - x^2}$, so the surface area is

$$S(x) = 2\pi\left(r^2 - x^2\right) + 4\pi x\sqrt{r^2 - x^2}, \quad 0 \le x \le r$$
$$= 2\pi r^2 - 2\pi x^2 + 4\pi\left(x\sqrt{r^2 - x^2}\right)$$

Thus, $S'(x) = 0 - 4\pi x + 4\pi\left[x \cdot \frac{1}{2}\left(r^2 - x^2\right)^{-1/2}(-2x) + \left(r^2 - x^2\right)^{1/2} \cdot 1\right]$

$$= 4\pi\left[-x - \frac{x^2}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2}\right] = 4\pi \cdot \frac{-x\sqrt{r^2 - x^2} - x^2 + r^2 - x^2}{\sqrt{r^2 - x^2}}$$

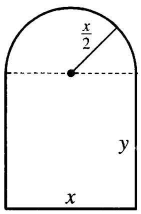
$S'(x) = 0 \Rightarrow x\sqrt{r^2 - x^2} = r^2 - 2x^2 \quad (\star) \Rightarrow \left(x\sqrt{r^2 - x^2}\right)^2 = \left(r^2 - 2x^2\right)^2 \Rightarrow$

$x^2(r^2 - x^2) = r^4 - 4r^2x^2 + 4x^4 \;\Rightarrow\; r^2x^2 - x^4 = r^4 - 4r^2x^2 + 4x^4 \;\Rightarrow\; 5x^4 - 5r^2x^2 + r^4 = 0.$

This is a quadratic equation in x^2. By the quadratic formula, $x^2 = \frac{5 \pm \sqrt{5}}{10} r^2$, but we reject the root with the + sign since it doesn't satisfy ($\star$). [The right side is negative and the left side is positive.] So $x = \sqrt{\frac{5 - \sqrt{5}}{10}}\, r$. Since $S(0) = S(r) = 0$, the maximum surface area occurs at the critical number and $x^2 = \frac{5 - \sqrt{5}}{10} r^2 \;\Rightarrow$ $y^2 = r^2 - \frac{5 - \sqrt{5}}{10} r^2 = \frac{5 + \sqrt{5}}{10} r^2 \;\Rightarrow$ the surface area is

$2\pi\left(\frac{5+\sqrt{5}}{10}\right)r^2 + 4\pi\sqrt{\frac{5-\sqrt{5}}{10}}\sqrt{\frac{5+\sqrt{5}}{10}}\,r^2 = \pi r^2\left[2 \cdot \frac{5+\sqrt{5}}{10} + 4\frac{\sqrt{(5-\sqrt{5})(5+\sqrt{5})}}{10}\right] = \pi r^2\left[\frac{5+\sqrt{5}}{5} + \frac{2\sqrt{20}}{5}\right] =$
$\pi r^2\left[\frac{5+\sqrt{5}+2\cdot 2\sqrt{5}}{5}\right] = \pi r^2\left[\frac{5+5\sqrt{5}}{5}\right] = \pi r^2\left(1 + \sqrt{5}\right).$

28.



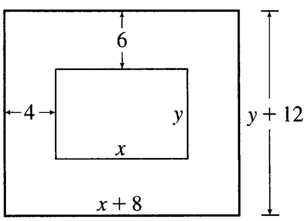
Perimeter $= 30 \;\Rightarrow\; 2y + x + \pi\left(\frac{x}{2}\right) = 30 \;\Rightarrow$

$y = \frac{1}{2}\left(30 - x - \frac{\pi x}{2}\right) = 15 - \frac{x}{2} - \frac{\pi x}{4}$. The area is the area of the rectangle plus the area of the semicircle, or $xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$, so

$A(x) = x\left(15 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{1}{8}\pi x^2 = 15x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2.$

$A'(x) = 15 - \left(1 + \frac{\pi}{4}\right)x = 0 \;\Rightarrow\; x = \dfrac{15}{1 + \pi/4} = \dfrac{60}{4 + \pi}$. $A''(x) = -\left(1 + \dfrac{\pi}{4}\right) < 0$, so this gives a maximum. The dimensions are $x = \dfrac{60}{4+\pi}$ ft and $y = 15 - \dfrac{30}{4+\pi} - \dfrac{15\pi}{4+\pi} = \dfrac{60 + 15\pi - 30 - 15\pi}{4+\pi} = \dfrac{30}{4+\pi}$ ft, so the height of the rectangle is half the base.

29.

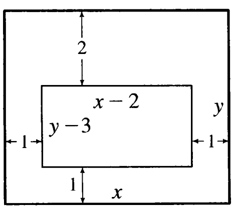


$xy = 384 \;\Rightarrow\; y = 384/x$. Total area is

$A(x) = (8 + x)(12 + 384/x) = 12(40 + x + 256/x)$, so

$A'(x) = 12\left(1 - 256/x^2\right) = 0 \;\Rightarrow\; x = 16$. There is an absolute minimum when $x = 16$ since $A'(x) < 0$ for $0 < x < 16$ and $A'(x) > 0$ for $x > 16$. When $x = 16$, $y = 384/16 = 24$, so the dimensions are 24 cm and 36 cm.

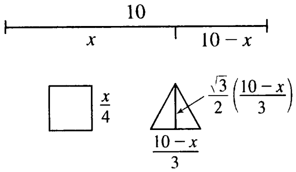
30.



$xy = 180$, so $y = 180/x$. The printed area is

$(x - 2)(y - 3) = (x - 2)(180/x - 3) = 186 - 3x - 360/x = A(x)$.

$A'(x) = -3 + 360/x^2 = 0$ when $x^2 = 120 \;\Rightarrow\; x = 2\sqrt{30}$. This gives an absolute maximum since $A'(x) > 0$ for $0 < x < 2\sqrt{30}$ and $A'(x) < 0$ for $x > 2\sqrt{30}$. When $x = 2\sqrt{30}$, $y = 180/(2\sqrt{30})$, so the dimensions are $2\sqrt{30}$ in. and $90/\sqrt{30}$ in.

31. 

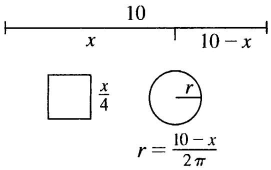
Let x be the length of the wire used for the square. The total area is

$$A(x) = \left(\frac{x}{4}\right)^2 + \frac{1}{2}\left(\frac{10-x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{10-x}{3}\right)$$
$$= \tfrac{1}{16}x^2 + \tfrac{\sqrt{3}}{36}(10-x)^2, \quad 0 \le x \le 10$$

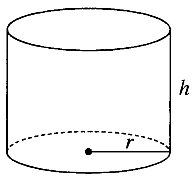
$A'(x) = \frac{1}{8}x - \frac{\sqrt{3}}{18}(10-x) = 0 \quad \Leftrightarrow \quad \frac{9}{72}x + \frac{4\sqrt{3}}{72}x - \frac{40\sqrt{3}}{72} = 0 \quad \Leftrightarrow \quad x = \frac{40\sqrt{3}}{9+4\sqrt{3}}$. Now $A(0) = \left(\frac{\sqrt{3}}{36}\right)100 \approx 4.81$, $A(10) = \frac{100}{16} = 6.25$ and $A\left(\frac{40\sqrt{3}}{9+4\sqrt{3}}\right) \approx 2.72$, so

(a) The maximum area occurs when $x = 10$ m, and all the wire is used for the square.

(b) The minimum area occurs when $x = \frac{40\sqrt{3}}{9+4\sqrt{3}} \approx 4.35$ m.

32. 

Total area is $A(x) = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{10-x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{(10-x)^2}{4\pi}$, $0 \le x \le 10$. $A'(x) = \frac{x}{8} - \frac{10-x}{2\pi} = \left(\frac{1}{2\pi} + \frac{1}{8}\right)x - \frac{5}{\pi} = 0 \quad \Rightarrow$ $x = 40/(4+\pi)$. $A(0) = 25/\pi \approx 7.96$, $A(10) = 6.25$, and $A(40/(4+\pi)) \approx 3.5$, so the maximum occurs when $x = 0$ m and the minimum occurs when $x = 40/(4+\pi)$ m.

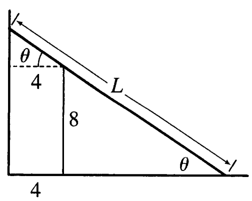
33. 

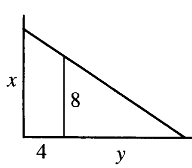
The volume is $V = \pi r^2 h$ and the surface area is

$$S(r) = \pi r^2 + 2\pi r h = \pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right) = \pi r^2 + \frac{2V}{r}.$$

$$S'(r) = 2\pi r - \frac{2V}{r^2} = 0 \quad \Rightarrow \quad 2\pi r^3 = 2V \quad \Rightarrow \quad r = \sqrt[3]{\frac{V}{\pi}} \text{ cm.}$$

This gives an absolute minimum since $S'(r) < 0$ for $0 < r < \sqrt[3]{\frac{V}{\pi}}$ and $S'(r) > 0$ for $r > \sqrt[3]{\frac{V}{\pi}}$. When $r = \sqrt[3]{\frac{V}{\pi}}$, $h = \frac{V}{\pi r^2} = \frac{V}{\pi (V/\pi)^{2/3}} = \sqrt[3]{\frac{V}{\pi}}$ cm.

34. 



$L = 8\csc\theta + 4\sec\theta$, $0 < \theta < \frac{\pi}{2}$.

$\frac{dL}{d\theta} = -8\csc\theta\,\cot\theta + 4\sec\theta\,\tan\theta = 0$ when

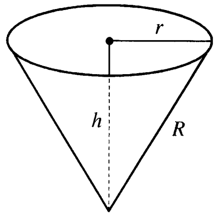
$\sec\theta\,\tan\theta = 2\csc\theta\,\cot\theta \quad \Leftrightarrow \quad \tan^3\theta = 2 \quad \Leftrightarrow \quad \tan\theta = \sqrt[3]{2} \quad \Leftrightarrow$ $\theta = \tan^{-1}\sqrt[3]{2}$.

$dL/d\theta < 0$ when $0 < \theta < \tan^{-1}\sqrt[3]{2}$, $dL/d\theta > 0$ when $\tan^{-1}\sqrt[3]{2} < \theta < \frac{\pi}{2}$, so L has an absolute minimum when $\theta = \tan^{-1}\sqrt[3]{2}$, and the shortest ladder has length

$$L = 8\frac{\sqrt{1+2^{2/3}}}{2^{1/3}} + 4\sqrt{1+2^{2/3}} \approx 16.65 \text{ ft.}$$

Another method: Minimize $L^2 = x^2 + (4+y)^2$, where $\frac{x}{4+y} = \frac{8}{y}$.

35.



$h^2 + r^2 = R^2 \Rightarrow V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}(R^2 - h^2)h = \frac{\pi}{3}(R^2 h - h^3)$.

$V'(h) = \frac{\pi}{3}(R^2 - 3h^2) = 0$ when $h = \frac{1}{\sqrt{3}}R$. This gives an absolute maximum, since $V'(h) > 0$ for $0 < h < \frac{1}{\sqrt{3}}R$ and $V'(h) < 0$ for $h > \frac{1}{\sqrt{3}}R$. The maximum volume is

$V\left(\frac{1}{\sqrt{3}}R\right) = \frac{\pi}{3}\left(\frac{1}{\sqrt{3}}R^3 - \frac{1}{3\sqrt{3}}R^3\right) = \frac{2}{9\sqrt{3}}\pi R^3$.

36. The volume and surface area of a cone with radius r and height h are given by $V = \frac{1}{3}\pi r^2 h$ and $S = \pi r\sqrt{r^2 + h^2}$.

We'll minimize $A = S^2$ subject to $V = 27$. $V = 27 \Rightarrow \frac{1}{3}\pi r^2 h = 27 \Rightarrow r^2 = \dfrac{81}{\pi h}$ **(1)**.

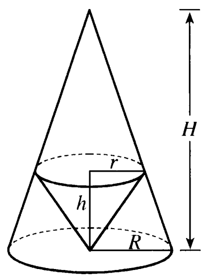
$$A = \pi^2 r^2 (r^2 + h^2) = \pi^2\left(\frac{81}{\pi h}\right)\left(\frac{81}{\pi h} + h^2\right) = \frac{81^2}{h^2} + 81\pi h, \text{ so } A' = 0 \Rightarrow \frac{-2\cdot 81^2}{h^3} + 81\pi = 0 \Rightarrow$$

$$81\pi = \frac{2\cdot 81^2}{h^3} \Rightarrow h^3 = \frac{162}{\pi} \Rightarrow h = \sqrt[3]{\frac{162}{\pi}} = 3\sqrt[3]{\frac{6}{\pi}} \approx 3.722. \text{ From } \mathbf{(1)},$$

$$r^2 = \frac{81}{\pi h} = \frac{81}{\pi\cdot 3\sqrt[3]{6/\pi}} = \frac{27}{\sqrt[3]{6\pi^2}} \Rightarrow r = \frac{3\sqrt{3}}{\sqrt[6]{6\pi^2}} \approx 2.632.\ A'' = 6\cdot 81^2/h^4 > 0, \text{ so } A \text{ and hence } S \text{ has an}$$

absolute minimum at these values of r and h.

37.



By similar triangles, $\dfrac{H}{R} = \dfrac{H-h}{r}$ **(1)**. The volume of the inner cone is $V = \frac{1}{3}\pi r^2 h$, so we'll solve **(1)** for h. $\dfrac{Hr}{R} = H - h \Rightarrow$

$$h = H - \frac{Hr}{R} = \frac{HR - Hr}{R} = \frac{H}{R}(R - r) \quad \mathbf{(2)}.$$

Thus, $V(r) = \dfrac{\pi}{3} r^2 \cdot \dfrac{H}{R}(R - r) = \dfrac{\pi H}{3R}(Rr^2 - r^3) \Rightarrow$

$$V'(r) = \frac{\pi H}{3R}(2Rr - 3r^2) = \frac{\pi H}{3R} r(2R - 3r).$$

$V'(r) = 0 \Rightarrow r = 0$ or $2R = 3r \Rightarrow r = \frac{2}{3}R$ and from **(2)**, $h = \dfrac{H}{R}\left(R - \frac{2}{3}R\right) = \dfrac{H}{R}\left(\frac{1}{3}R\right) = \frac{1}{3}H$.

$V'(r)$ changes from positive to negative at $r = \frac{2}{3}R$, so the inner cone has a maximum volume of

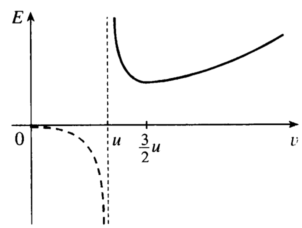
$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{3}R\right)^2\left(\frac{1}{3}H\right) = \frac{4}{27}\cdot\frac{1}{3}\pi R^2 H$, which is approximately 15% of the volume of the larger cone.

38. (a) $E(v) = \dfrac{aLv^3}{v - u} \Rightarrow E'(v) = aL\dfrac{(v-u)3v^2 - v^3}{(v-u)^2} = 0$

when $2v^3 = 3uv^2 \Rightarrow 2v = 3u \Rightarrow v = \frac{3}{2}u$.

The First Derivative Test shows that this value of v gives the minimum value of E.

(b)

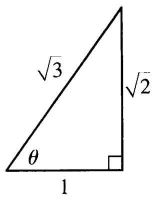


39. $S = 6sh - \frac{3}{2}s^2\cot\theta + 3s^2\frac{\sqrt{3}}{2}\csc\theta$

(a) $\dfrac{dS}{d\theta} = \frac{3}{2}s^2\csc^2\theta - 3s^2\frac{\sqrt{3}}{2}\csc\theta\,\cot\theta$ or $\frac{3}{2}s^2\csc\theta\left(\csc\theta - \sqrt{3}\cot\theta\right)$.

(b) $\dfrac{dS}{d\theta} = 0$ when $\csc\theta - \sqrt{3}\cot\theta = 0 \;\Rightarrow\; \dfrac{1}{\sin\theta} - \sqrt{3}\,\dfrac{\cos\theta}{\sin\theta} = 0 \;\Rightarrow\; \cos\theta = \frac{1}{\sqrt{3}}$. The First Derivative Test shows that the minimum surface area occurs when $\theta = \cos^{-1}\!\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ$.

(c)

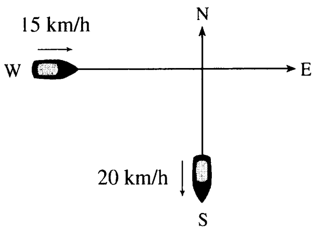


If $\cos\theta = \frac{1}{\sqrt{3}}$, then $\cot\theta = \frac{1}{\sqrt{2}}$ and $\csc\theta = \frac{\sqrt{3}}{\sqrt{2}}$, so the surface area is

$$S = 6sh - \tfrac{3}{2}s^2\tfrac{1}{\sqrt{2}} + 3s^2\tfrac{\sqrt{3}}{2}\tfrac{\sqrt{3}}{\sqrt{2}} = 6sh - \tfrac{3}{2\sqrt{2}}s^2 + \tfrac{9}{2\sqrt{2}}s^2$$

$$= 6sh + \tfrac{6}{2\sqrt{2}}s^2 = 6s\left(h + \tfrac{1}{2\sqrt{2}}s\right)$$

40.

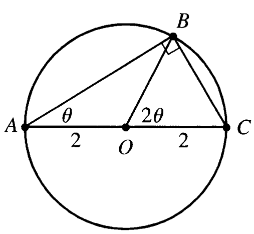


Let t be the time, in hours, after 2:00 P.M. The position of the boat heading south at time t is $(0, -20t)$. The position of the boat heading east at time t is $(-15 + 15t, 0)$. If $D(t)$ is the distance between the boats at time t, we minimize $f(t) = [D(t)]^2 = 20^2t^2 + 15^2(t-1)^2$.

$f'(t) = 800t + 450(t-1) = 1250t - 450 = 0$ when $t = \frac{450}{1250} = 0.36$ h. $0.36\text{ h} \times \frac{60\text{ min}}{\text{h}} = 21.6\text{ min} = 21\text{ min }36\text{ s}$. Since $f''(t) > 0$, this gives a minimum, so the boats are closest together at 2:21:36 P.M.

41. Here $T(x) = \dfrac{\sqrt{x^2+25}}{6} + \dfrac{5-x}{8}$, $0 \le x \le 5 \;\Rightarrow\; T'(x) = \dfrac{x}{6\sqrt{x^2+25}} - \dfrac{1}{8} = 0 \;\Leftrightarrow\; 8x = 6\sqrt{x^2+25}$ $\Leftrightarrow\; 16x^2 = 9(x^2+25) \;\Leftrightarrow\; x = \frac{15}{\sqrt{7}}$. But $\frac{15}{\sqrt{7}} > 5$, so T has no critical number. Since $T(0) \approx 1.46$ and $T(5) \approx 1.18$, he should row directly to B.

42.



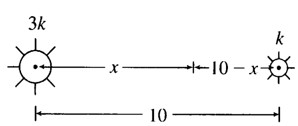
In isosceles triangle AOB, $\angle O = 180^\circ - \theta - \theta$, so $\angle BOC = 2\theta$. The distance rowed is $4\cos\theta$ while the distance walked is the length of arc $BC = 2(2\theta) = 4\theta$. The time taken is given by

$$T(\theta) = \frac{4\cos\theta}{2} + \frac{4\theta}{4} = 2\cos\theta + \theta,\quad 0 \le \theta \le \tfrac{\pi}{2}.$$

$$T'(\theta) = -2\sin\theta + 1 = 0 \;\Leftrightarrow\; \sin\theta = \tfrac{1}{2} \;\Rightarrow\; \theta = \tfrac{\pi}{6}.$$

Check the value of T at $\theta = \frac{\pi}{6}$ and at the endpoints of the domain of T; that is, $\theta = 0$ and $\theta = \frac{\pi}{2}$. $T(0) = 2$, $T\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{\pi}{6} \approx 2.26$, and $T\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \approx 1.57$. Therefore, the minimum value of T is $\frac{\pi}{2}$ when $\theta = \frac{\pi}{2}$; that is, the woman should walk all the way. Note that $T''(\theta) = -2\cos\theta < 0$ for $0 \le \theta < \frac{\pi}{2}$, so $\theta = \frac{\pi}{6}$ gives a maximum time.

43.



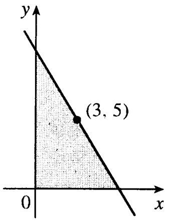
The total illumination is $I(x) = \dfrac{3k}{x^2} + \dfrac{k}{(10-x)^2}$, $0 < x < 10$. Then

$$I'(x) = \frac{-6k}{x^3} + \frac{2k}{(10-x)^3} = 0 \;\Rightarrow\; 6k(10-x)^3 = 2kx^3 \;\Rightarrow$$

$$3(10-x)^3 = x^3 \;\Rightarrow\; \sqrt[3]{3}\,(10-x) = x \;\Rightarrow\; 10\sqrt[3]{3} - \sqrt[3]{3}\,x = x$$

$$\Rightarrow\; 10\sqrt[3]{3} = x + \sqrt[3]{3}\,x \;\Rightarrow\; 10\sqrt[3]{3} = \left(1 + \sqrt[3]{3}\right)x \;\Rightarrow$$

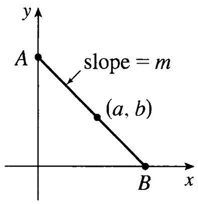
$x = \dfrac{10\sqrt[3]{3}}{1+\sqrt[3]{3}} \approx 5.9$ ft. This gives a minimum since $I''(x) > 0$ for $0 < x < 10$.

44. 

The line with slope m (where $m < 0$) through $(3, 5)$ has equation $y - 5 = m(x - 3)$ or $y = mx + (5 - 3m)$. The y-intercept is $5 - 3m$ and the x-intercept is $-5/m + 3$. So the triangle has area $A(m) = \frac{1}{2}(5 - 3m)(-5/m + 3) = 15 - 25/(2m) - \frac{9}{2}m$.

Now $A'(m) = \dfrac{25}{2m^2} - \dfrac{9}{2} = 0 \quad \Leftrightarrow \quad m^2 = \frac{25}{9} \quad \Rightarrow \quad m = -\frac{5}{3}$ (since $m < 0$).

$A''(m) = -\dfrac{25}{m^3} > 0$, so there is an absolute minimum when $m = -\frac{5}{3}$. Thus, an equation of the line is $y - 5 = -\frac{5}{3}(x - 3)$ or $y = -\frac{5}{3}x + 10$.

45. 

Every line segment in the first quadrant passing through (a, b) with endpoints on the x- and y-axes satisfies an equation of the form $y - b = m(x - a)$, where $m < 0$. By setting $x = 0$ and then $y = 0$, we find its endpoints, $A(0, b - am)$ and $B\left(a - \frac{b}{m}, 0\right)$. The distance d from A to B is given by $d = \sqrt{[(a - \frac{b}{m}) - 0]^2 + [0 - (b - am)]^2}$.

It follows that the square of the length of the line segment, as a function of m, is given by

$$S(m) = \left(a - \frac{b}{m}\right)^2 + (am - b)^2 = a^2 - \frac{2ab}{m} + \frac{b^2}{m^2} + a^2m^2 - 2abm + b^2. \text{ Thus,}$$

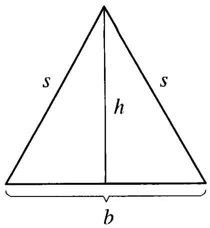
$$\begin{aligned} S'(m) &= \frac{2ab}{m^2} - \frac{2b^2}{m^3} + 2a^2m - 2ab = \frac{2}{m^3}(abm - b^2 + a^2m^4 - abm^3) \\ &= \frac{2}{m^3}[b(am - b) + am^3(am - b)] = \frac{2}{m^3}(am - b)(b + am^3) \end{aligned}$$

Thus, $S'(m) = 0 \quad \Leftrightarrow \quad m = b/a$ or $m = -\sqrt[3]{\frac{b}{a}}$. Since $b/a > 0$ and $m < 0$, m must equal $-\sqrt[3]{\frac{b}{a}}$. Since $\dfrac{2}{m^3} < 0$, we see that $S'(m) < 0$ for $m < -\sqrt[3]{\frac{b}{a}}$ and $S'(m) > 0$ for $m > -\sqrt[3]{\frac{b}{a}}$. Thus, S has its absolute minimum value when $m = -\sqrt[3]{\frac{b}{a}}$. That value is

$$\begin{aligned} S\left(-\sqrt[3]{\tfrac{b}{a}}\right) &= \left(a + b\sqrt[3]{\tfrac{a}{b}}\right)^2 + \left(-a\sqrt[3]{\tfrac{b}{a}} - b\right)^2 = \left(a + \sqrt[3]{ab^2}\right)^2 + \left(\sqrt[3]{a^2b} + b\right)^2 \\ &= a^2 + 2a^{4/3}b^{2/3} + a^{2/3}b^{4/3} + a^{4/3}b^{2/3} + 2a^{2/3}b^{4/3} + b^2 = a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2 \end{aligned}$$

The last expression is of the form $x^3 + 3x^2y + 3xy^2 + y^3 \quad [= (x + y)^3]$ with $x = a^{2/3}$ and $y = b^{2/3}$, so we can write it as $(a^{2/3} + b^{2/3})^3$ and the shortest such line segment has length $\sqrt{S} = (a^{2/3} + b^{2/3})^{3/2}$.

46. $y = 1 + 40x^3 - 3x^5 \quad \Rightarrow \quad y' = 120x^2 - 15x^4$, so the tangent line to the curve at $x = a$ has slope $m(a) = 120a^2 - 15a^4$. Now $m'(a) = 240a - 60a^3 = -60a(a^2 - 4) = -60a(a + 2)(a - 2)$, so $m'(a) > 0$ for $a < -2$, and $0 < a < 2$, and $m'(a) < 0$ for $-2 < a < 0$ and $a > 2$. Thus, m is increasing on $(-\infty, -2)$, decreasing on $(-2, 0)$, increasing on $(0, 2)$, and decreasing on $(2, \infty)$. Clearly, $m(a) \to -\infty$ as $a \to \pm\infty$, so the maximum value of $m(a)$ must be one of the two local maxima, $m(-2)$ or $m(2)$. But both $m(-2)$ and $m(2)$ equal $120 \cdot 2^2 - 15 \cdot 2^4 = 480 - 240 = 240$. So 240 is the largest slope, and it occurs at the points $(-2, -223)$ and $(2, 225)$. *Note:* $a = 0$ corresponds to a local *minimum* of m.

47. 

Here $s^2 = h^2 + b^2/4$, so $h^2 = s^2 - b^2/4$. The area is $A = \frac{1}{2}b\sqrt{s^2 - b^2/4}$.

Let the perimeter be p, so $2s + b = p$ or $s = (p-b)/2 \Rightarrow$

$A(b) = \frac{1}{2}b\sqrt{(p-b)^2/4 - b^2/4} = b\sqrt{p^2 - 2pb}/4$. Now

$A'(b) = \dfrac{\sqrt{p^2-2pb}}{4} - \dfrac{bp/4}{\sqrt{p^2-2pb}} = \dfrac{-3pb+p^2}{4\sqrt{p^2-2pb}}$. Therefore, $A'(b) = 0 \Rightarrow$

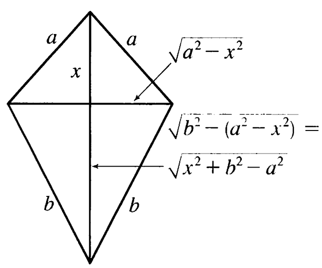
$-3pb + p^2 = 0 \Rightarrow b = p/3$. Since $A'(b) > 0$ for $b < p/3$ and $A'(b) < 0$ for $b > p/3$, there is an absolute maximum when $b = p/3$. But then $2s + p/3 = p$, so $s = p/3 \Rightarrow s = b \Rightarrow$ the triangle is equilateral.

48. See the figure. The area is given by

$A(x) = \frac{1}{2}\left(2\sqrt{a^2-x^2}\right)x + \frac{1}{2}\left(2\sqrt{a^2-x^2}\right)\left(\sqrt{x^2+b^2-a^2}\right) = \sqrt{a^2-x^2}\left(x + \sqrt{x^2+b^2-a^2}\right)$ for

$0 \le x \le a$. Now $A'(x) = \sqrt{a^2-x^2}\left(1 + \dfrac{x}{\sqrt{x^2+b^2-a^2}}\right) + \left(x + \sqrt{x^2+b^2-a^2}\right)\dfrac{-x}{\sqrt{a^2-x^2}} = 0 \Leftrightarrow$

$$\frac{x}{\sqrt{a^2-x^2}}\left(x + \sqrt{x^2+b^2-a^2}\right) = \sqrt{a^2-x^2}\left(\frac{x + \sqrt{x^2+b^2-a^2}}{\sqrt{x^2+b^2-a^2}}\right).$$



Except for the trivial case where $x = 0$, $a = b$ and $A(x) = 0$, we have $x + \sqrt{x^2+b^2-a^2} > 0$. Hence, cancelling this factor gives

$\dfrac{x}{\sqrt{a^2-x^2}} = \dfrac{\sqrt{a^2-x^2}}{\sqrt{x^2+b^2-a^2}} \Rightarrow x\sqrt{x^2+b^2-a^2} = a^2 - x^2 \Rightarrow$

$x^2\left(x^2+b^2-a^2\right) = a^4 - 2a^2x^2 + x^4 \Rightarrow x^2\left(b^2-a^2\right) = a^4 - 2a^2x^2$

$\Rightarrow x^2\left(b^2+a^2\right) = a^4 \Rightarrow x = \dfrac{a^2}{\sqrt{a^2+b^2}}$.

Now we must check the value of A at this point as well as at the endpoints of the domain to see which gives the maximum value. $A(0) = a\sqrt{b^2-a^2}$, $A(a) = 0$ and

$$A\left(\frac{a^2}{\sqrt{a^2+b^2}}\right) = \sqrt{a^2 - \left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2}\left[\frac{a^2}{\sqrt{a^2+b^2}} + \sqrt{\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2 + b^2 - a^2}\right]$$

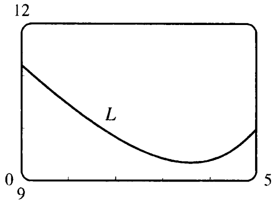
$$= \frac{ab}{\sqrt{a^2+b^2}}\left[\frac{a^2}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{a^2+b^2}}\right] = \frac{ab\left(a^2+b^2\right)}{a^2+b^2} = ab$$

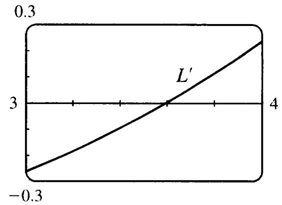
Since $b \ge \sqrt{b^2-a^2}$, $A\left(a^2/\sqrt{a^2+b^2}\right) \ge A(0)$. So there is an absolute maximum when $x = \dfrac{a^2}{\sqrt{a^2+b^2}}$. In this case the horizontal piece should be $\dfrac{2ab}{\sqrt{a^2+b^2}}$ and the vertical piece should be $\dfrac{a^2+b^2}{\sqrt{a^2+b^2}} = \sqrt{a^2+b^2}$.

49. Note that $|AD| = |AP| + |PD| \quad\Rightarrow\quad 5 = x + |PD| \quad\Rightarrow\quad |PD| = 5 - x$. Using the Pythagorean Theorem for ΔPDB and ΔPDC gives us

$$L(x) = |AP| + |BP| + |CP| = x + \sqrt{(5-x)^2 + 2^2} + \sqrt{(5-x)^2 + 3^2}$$
$$= x + \sqrt{x^2 - 10x + 29} + \sqrt{x^2 - 10x + 34}$$

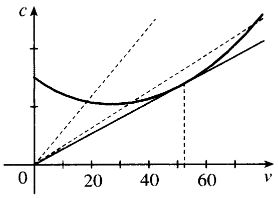
$$\Rightarrow\quad L'(x) = 1 + \frac{x-5}{\sqrt{x^2 - 10x + 29}} + \frac{x-5}{\sqrt{x^2 - 10x + 34}}.$$





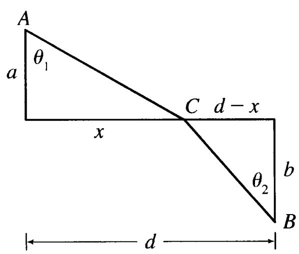
From the graphs of L and L', it seems that the minimum value of L is about $L(3.59) = 9.35$ m.

50. We note that since c is the consumption in gallons per hour, and v is the velocity in miles per hour, then $\dfrac{c}{v} = \dfrac{\text{gallons/hour}}{\text{miles/hour}} = \dfrac{\text{gallons}}{\text{mile}}$ gives us the consumption in gallons per mile, that is, the quantity G. To find the minimum, we calculate $\dfrac{dG}{dv} = \dfrac{d}{dv}\left(\dfrac{c}{v}\right) = \dfrac{v\dfrac{dc}{dv} - c\dfrac{dv}{dv}}{v^2} = \dfrac{v\dfrac{dc}{dv} - c}{v^2}$.



This is 0 when $v\dfrac{dc}{dv} - c = 0 \quad\Leftrightarrow\quad \dfrac{dc}{dv} = \dfrac{c}{v}$. This implies that the tangent line of $c(v)$ passes through the origin, and this occurs when $v \approx 53$ mi/h. Note that the slope of the secant line through the origin and a point $(v, c(v))$ on the graph is equal to $G(v)$, and it is intuitively clear that G is minimized in the case where the secant is in fact a tangent.

51.



The total time is

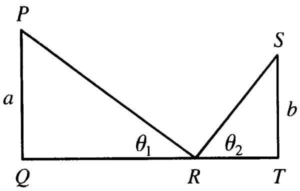
$$T(x) = (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B)$$
$$= \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (d-x)^2}}{v_2}, \quad 0 < x < d$$

$$T'(x) = \frac{x}{v_1\sqrt{a^2 + x^2}} - \frac{d-x}{v_2\sqrt{b^2 + (d-x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

The minimum occurs when $T'(x) = 0 \quad\Rightarrow\quad \dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$.

[*Note:* $T''(x) > 0$]

52.



If $d = |QT|$, we minimize $f(\theta_1) = |PR| + |RS| = a\csc\theta_1 + b\csc\theta_2$. Differentiating with respect to θ_1, and setting $\dfrac{df}{d\theta_1}$ equal to 0, we get

$$\frac{df}{d\theta_1} = 0 = -a\csc\theta_1\cot\theta_1 - b\csc\theta_2\cot\theta_2\,\frac{d\theta_2}{d\theta_1}.$$

So we need to find an expression for $\dfrac{d\theta_2}{d\theta_1}$. We can do this by observing that $|QT| = \text{constant} = a\cot\theta_1 + b\cot\theta_2$.

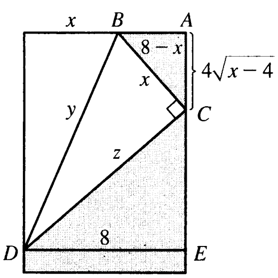
Differentiating this equation implicitly with respect to θ_1, we get $-a\csc^2\theta_1 - b\csc^2\theta_2\,\dfrac{d\theta_2}{d\theta_1} = 0 \;\Rightarrow$

$\dfrac{d\theta_2}{d\theta_1} = -\dfrac{a\csc^2\theta_1}{b\csc^2\theta_2}$. We substitute this into the expression for $\dfrac{df}{d\theta_1}$ to get

$$-a\csc\theta_1\cot\theta_1 - b\csc\theta_2\cot\theta_2\left(-\frac{a\csc^2\theta_1}{b\csc^2\theta_2}\right) = 0 \;\Leftrightarrow\; -a\csc\theta_1\cot\theta_1 + a\frac{\csc^2\theta_1\cot\theta_2}{\csc\theta_2} = 0 \;\Leftrightarrow$$

$\cot\theta_1\csc\theta_2 = \csc\theta_1\cot\theta_2 \;\Leftrightarrow\; \dfrac{\cot\theta_1}{\csc\theta_1} = \dfrac{\cot\theta_2}{\csc\theta_2} \;\Leftrightarrow\; \cos\theta_1 = \cos\theta_2$. Since θ_1 and θ_2 are both acute, we have $\theta_1 = \theta_2$.

53.



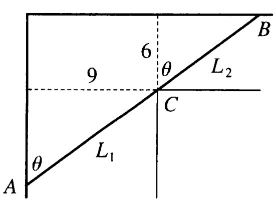
$y^2 = x^2 + z^2$, but triangles CDE and BCA are similar, so $z/8 = x/\big(4\sqrt{x-4}\big) \;\Rightarrow\; z = 2x/\sqrt{x-4}$. Thus, we minimize

$f(x) = y^2 = x^2 + 4x^2/(x-4) = x^3/(x-4),\quad 4 < x \le 8.$

$$f'(x) = \frac{(x-4)(3x^2) - x^3}{(x-4)^2} = \frac{x^2[3(x-4) - x]}{(x-4)^2} = \frac{2x^2(x-6)}{(x-4)^2} = 0$$

when $x = 6$. $f'(x) < 0$ when $x < 6$, $f'(x) > 0$ when $x > 6$, so the minimum occurs when $x = 6$ in.

54.

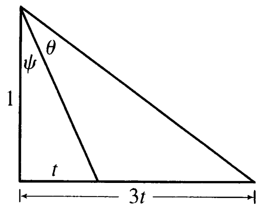


Paradoxically, we solve this maximum problem by solving a minimum problem. Let L be the length of the line ACB going from wall to wall touching the inner corner C. As $\theta \to 0$ or $\theta \to \frac{\pi}{2}$, we have $L \to \infty$ and there will be an angle that makes L a minimum. A pipe of this length will just fit around the corner.

From the diagram, $L = L_1 + L_2 = 9\csc\theta + 6\sec\theta \;\Rightarrow\; dL/d\theta = -9\csc\theta\,\cot\theta + 6\sec\theta\,\tan\theta = 0$ when $6\sec\theta\,\tan\theta = 9\csc\theta\,\cot\theta \;\Leftrightarrow\; \tan^3\theta = \frac{9}{6} = 1.5 \;\Leftrightarrow\; \tan\theta = \sqrt[3]{1.5}$. Then $\sec^2\theta = 1 + \left(\frac{3}{2}\right)^{2/3}$ and $\csc^2\theta = 1 + \left(\frac{3}{2}\right)^{-2/3}$, so the longest pipe has length $L = 9\left[1 + \left(\frac{3}{2}\right)^{-2/3}\right]^{1/2} + 6\left[1 + \left(\frac{3}{2}\right)^{2/3}\right]^{1/2} \approx 21.07$ ft.

Or, use $\theta = \tan^{-1}\big(\sqrt[3]{1.5}\big) \approx 0.852 \;\Rightarrow\; L = 9\csc\theta + 6\sec\theta \approx 21.07$ ft.

55.



It suffices to maximize $\tan\theta$. Now

$$\frac{3t}{1} = \tan(\psi+\theta) = \frac{\tan\psi + \tan\theta}{1 - \tan\psi\,\tan\theta} = \frac{t+\tan\theta}{1 - t\tan\theta}.$$

So $3t(1 - t\tan\theta) = t + \tan\theta \;\Rightarrow\; 2t = \left(1+3t^2\right)\tan\theta \;\Rightarrow$

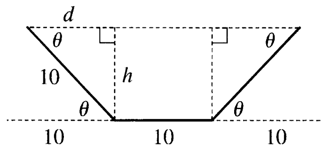
$\tan\theta = \dfrac{2t}{1+3t^2}$. Let $f(t) = \tan\theta = \dfrac{2t}{1+3t^2} \;\Rightarrow$

$$f'(t) = \frac{2\left(1+3t^2\right) - 2t(6t)}{(1+3t^2)^2} = \frac{2\left(1-3t^2\right)}{(1+3t^2)^2} = 0 \;\Leftrightarrow$$

$1 - 3t^2 = 0 \;\Leftrightarrow\; t = \frac{1}{\sqrt{3}}$ since $t \ge 0$.

Now $f'(t) > 0$ for $0 \le t < \frac{1}{\sqrt{3}}$ and $f'(t) < 0$ for $t > \frac{1}{\sqrt{3}}$, so f has an absolute maximum when $t = \frac{1}{\sqrt{3}}$ and $\tan\theta = \dfrac{2\left(1/\sqrt{3}\right)}{1+3\left(1/\sqrt{3}\right)^2} = \frac{1}{\sqrt{3}} \;\Rightarrow\; \theta = \frac{\pi}{6}$. Substituting for t and θ in $3t = \tan(\psi+\theta)$ gives us $\sqrt{3} = \tan\left(\psi + \frac{\pi}{6}\right) \;\Rightarrow\; \psi = \frac{\pi}{6}$.

56.

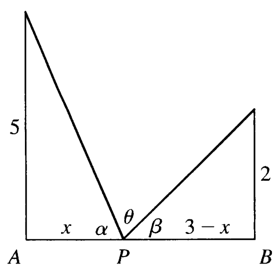


We maximize the cross-sectional area

$$\begin{aligned} A(\theta) &= 10h + 2\left(\tfrac{1}{2}dh\right) = 10h + dh = 10(10\sin\theta) + (10\cos\theta)(10\sin\theta) \\ &= 100(\sin\theta + \sin\theta\,\cos\theta),\quad 0 \le \theta \le \tfrac{\pi}{2} \\ A'(\theta) &= 100\left(\cos\theta + \cos^2\theta - \sin^2\theta\right) = 100\left(\cos\theta + 2\cos^2\theta - 1\right) = 100(2\cos\theta - 1)(\cos\theta + 1) \\ &= 0 \text{ when } \cos\theta = \tfrac{1}{2} \;\Leftrightarrow\; \theta = \tfrac{\pi}{3}. \quad \left(\cos\theta \ne -1 \text{ since } 0 \le \theta \le \tfrac{\pi}{2}.\right) \end{aligned}$$

Now $A(0) = 0$, $A\left(\frac{\pi}{2}\right) = 100$ and $A\left(\frac{\pi}{3}\right) = 75\sqrt{3} \approx 129.9$, so the maximum occurs when $\theta = \frac{\pi}{3}$.

57.



From the figure, $\tan\alpha = \dfrac{5}{x}$ and $\tan\beta = \dfrac{2}{3-x}$. Since

$\alpha + \beta + \theta = 180^\circ = \pi$, $\theta = \pi - \tan^{-1}\left(\dfrac{5}{x}\right) - \tan^{-1}\left(\dfrac{2}{3-x}\right) \;\Rightarrow$

$$\begin{aligned} \frac{d\theta}{dx} &= -\frac{1}{1+\left(\frac{5}{x}\right)^2}\left(-\frac{5}{x^2}\right) - \frac{1}{1+\left(\frac{2}{3-x}\right)^2}\left[\frac{2}{(3-x)^2}\right] \\ &= \frac{x^2}{x^2+25}\cdot\frac{5}{x^2} - \frac{(3-x)^2}{(3-x)^2+4}\cdot\frac{2}{(3-x)^2}. \end{aligned}$$

Now $\dfrac{d\theta}{dx} = 0 \;\Rightarrow\; \dfrac{5}{x^2+25} = \dfrac{2}{x^2 - 6x + 13} \;\Rightarrow\; 2x^2 + 50 = 5x^2 - 30x + 65 \;\Rightarrow$

$3x^2 - 30x + 15 = 0 \;\Rightarrow\; x^2 - 10x + 5 = 0 \;\Rightarrow\; x = 5 \pm 2\sqrt{5}$. We reject the root with the $+$ sign, since it is larger than 3. $d\theta/dx > 0$ for $x < 5 - 2\sqrt{5}$ and $d\theta/dx < 0$ for $x > 5 - 2\sqrt{5}$, so θ is maximized when $|AP| = x = 5 - 2\sqrt{5} \approx 0.53$.

58. Let x be the distance from the observer to the wall. Then, from the given figure,

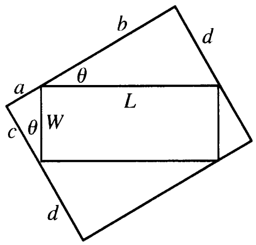
$$\theta = \tan^{-1}\left(\frac{h+d}{x}\right) - \tan^{-1}\left(\frac{d}{x}\right),\ x > 0 \quad \Rightarrow$$

$$\frac{d\theta}{dx} = \frac{1}{1+[(h+d)/x]^2}\left[-\frac{h+d}{x^2}\right] - \frac{1}{1+(d/x)^2}\left[-\frac{d}{x^2}\right] = -\frac{h+d}{x^2+(h+d)^2} + \frac{d}{x^2+d^2}$$

$$= \frac{d[x^2+(h+d)^2] - (h+d)(x^2+d^2)}{[x^2+(h+d)^2](x^2+d^2)} = \frac{h^2d+hd^2-hx^2}{[x^2+(h+d)^2](x^2+d^2)} = 0 \quad \Leftrightarrow$$

$hx^2 = h^2d + hd^2 \quad \Leftrightarrow \quad x^2 = hd + d^2 \quad \Leftrightarrow \quad x = \sqrt{d(h+d)}$. Since $d\theta/dx > 0$ for all $x < \sqrt{d(h+d)}$ and $d\theta/dx < 0$ for all $x > \sqrt{d(h+d)}$, the absolute maximum occurs when $x = \sqrt{d(h+d)}$.

59.



In the small triangle with sides a and c and hypotenuse W, $\sin\theta = \dfrac{a}{W}$ and $\cos\theta = \dfrac{c}{W}$. In the triangle with sides b and d and hypotenuse L, $\sin\theta = \dfrac{d}{L}$ and $\cos\theta = \dfrac{b}{L}$. Thus, $a = W\sin\theta$, $c = W\cos\theta$, $d = L\sin\theta$, and $b = L\cos\theta$, so the area of the circumscribed rectangle is

$$\begin{aligned}
A(\theta) &= (a+b)(c+d) = (W\sin\theta + L\cos\theta)(W\cos\theta + L\sin\theta) \\
&= W^2\sin\theta\,\cos\theta + WL\sin^2\theta + LW\cos^2\theta + L^2\sin\theta\,\cos\theta \\
&= LW\sin^2\theta + LW\cos^2\theta + (L^2+W^2)\sin\theta\,\cos\theta \\
&= LW(\sin^2\theta + \cos^2\theta) + (L^2+W^2)\cdot\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta \\
&= LW + \tfrac{1}{2}(L^2+W^2)\sin 2\theta, \quad 0 \le \theta \le \tfrac{\pi}{2}
\end{aligned}$$

This expression shows, without calculus, that the maximum value of $A(\theta)$ occurs when $\sin 2\theta = 1 \quad \Leftrightarrow \quad 2\theta = \frac{\pi}{2}$ $\Rightarrow \quad \theta = \frac{\pi}{4}$. So the maximum area is $A(\frac{\pi}{4}) = LW + \frac{1}{2}(L^2+W^2) = \frac{1}{2}(L^2+2LW+W^2) = \frac{1}{2}(L+W)^2$.

60. (a) Let D be the point such that $a = |AD|$. From the figure, $\sin\theta = \dfrac{b}{|BC|} \quad \Rightarrow \quad |BC| = b\csc\theta$ and

$\cos\theta = \dfrac{|BD|}{|BC|} = \dfrac{a-|AB|}{|BC|} \quad \Rightarrow \quad |BC| = (a-|AB|)\sec\theta$. Eliminating $|BC|$ gives

$(a-|AB|)\sec\theta = b\csc\theta \quad \Rightarrow \quad b\cot\theta = a - |AB| \quad \Rightarrow \quad |AB| = a - b\cot\theta$. The total resistance is

$$R(\theta) = C\frac{|AB|}{r_1^4} + C\frac{|BC|}{r_2^4} = C\left(\frac{a-b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right).$$

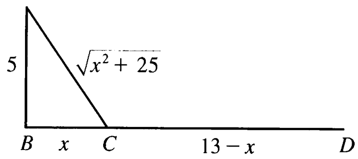
(b) $R'(\theta) = C\left(\dfrac{b\csc^2\theta}{r_1^4} - \dfrac{b\csc\theta\,\cot\theta}{r_2^4}\right) = bC\csc\theta\left(\dfrac{\csc\theta}{r_1^4} - \dfrac{\cot\theta}{r_2^4}\right)$.

$R'(\theta) = 0 \quad \Leftrightarrow \quad \dfrac{\csc\theta}{r_1^4} = \dfrac{\cot\theta}{r_2^4} \quad \Leftrightarrow \quad \dfrac{r_2^4}{r_1^4} = \dfrac{\cot\theta}{\csc\theta} = \cos\theta$.

$R'(\theta) > 0 \quad \Leftrightarrow \quad \dfrac{\csc\theta}{r_1^4} > \dfrac{\cot\theta}{r_2^4} \quad \Rightarrow \quad \cos\theta < \dfrac{r_2^4}{r_1^4}$ and $R'(\theta) < 0$ when $\cos\theta > \dfrac{r_2^4}{r_1^4}$, so there is an absolute minimum when $\cos\theta = r_2^4/r_1^4$.

(c) When $r_2 = \frac{2}{3}r_1$, we have $\cos\theta = \left(\frac{2}{3}\right)^4$, so $\theta = \cos^{-1}\left(\frac{2}{3}\right)^4 \approx 79^\circ$.

61. (a)



If $k =$ energy/km over land, then energy/km over water $= 1.4k$. So the total energy is $E = 1.4k\sqrt{25+x^2} + k(13-x)$, $0 \le x \le 13$, and so $\dfrac{dE}{dx} = \dfrac{1.4kx}{(25+x^2)^{1/2}} - k$.

Set $\dfrac{dE}{dx} = 0$: $1.4kx = k(25+x^2)^{1/2} \;\Rightarrow\; 1.96x^2 = x^2 + 25 \;\Rightarrow\; 0.96x^2 = 25 \;\Rightarrow\; x = \frac{5}{\sqrt{0.96}} \approx 5.1$. Testing against the value of E at the endpoints: $E(0) = 1.4k(5) + 13k = 20k$, $E(5.1) \approx 17.9k$, $E(13) \approx 19.5k$. Thus, to minimize energy, the bird should fly to a point about 5.1 km from B.

(b) If W/L is large, the bird would fly to a point C that is closer to B than to D to minimize the energy used flying over water. If W/L is small, the bird would fly to a point C that is closer to D than to B to minimize the distance of the flight. $E = W\sqrt{25+x^2} + L(13-x) \;\Rightarrow\; \dfrac{dE}{dx} = \dfrac{Wx}{\sqrt{25+x^2}} - L = 0$ when $\dfrac{W}{L} = \dfrac{\sqrt{25+x^2}}{x}$. By the same sort of argument as in part (a), this ratio will give the minimal expenditure of energy if the bird heads for the point x km from B.

(c) For flight direct to D, $x = 13$, so from part (b), $W/L = \frac{\sqrt{25+13^2}}{13} \approx 1.07$. There is no value of W/L for which the bird should fly directly to B. But note that $\lim\limits_{x\to 0^+}(W/L) = \infty$, so if the point at which E is a minimum is close to B, then W/L is large.

(d) Assuming that the birds instinctively choose the path that minimizes the energy expenditure, we can use the equation for $dE/dx = 0$ from part (a) with $1.4k = c$, $x = 4$, and $k = 1$: $(c)(4) = 1\cdot(25+4^2)^{1/2} \;\Rightarrow\; c = \sqrt{41}/4 \approx 1.6$.

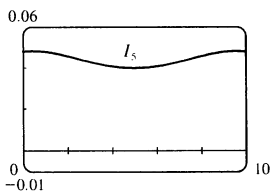
62. (a) $I(x) \propto \dfrac{\text{strength of source}}{(\text{distance from source})^2}$. Adding the intensities from the left and right lightbulbs,

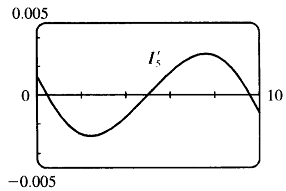
$$I(x) = \frac{k}{x^2+d^2} + \frac{k}{(10-x)^2+d^2} = \frac{k}{x^2+d^2} + \frac{k}{x^2-20x+100+d^2}.$$

(b) The magnitude of the constant k won't affect the location of the point of maximum intensity, so for convenience we take $k = 1$. $I'(x) = -\dfrac{2x}{(x^2+d^2)^2} - \dfrac{2(x-10)}{(x^2-20x+100+d^2)^2}$.

Substituting $d = 5$ into the equations for $I(x)$and $I'(x)$, we get

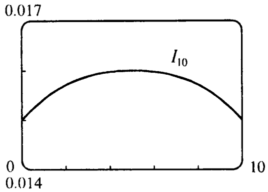
$$I_5(x) = \frac{1}{x^2+25} + \frac{1}{x^2-20x+125} \quad \text{and} \quad I_5'(x) = -\frac{2x}{(x^2+25)^2} - \frac{2(x-10)}{(x^2-20x+125)^2}$$

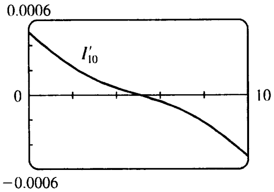




From the graphs, it appears that $I_5(x)$ has a minimum at $x = 5$ m.

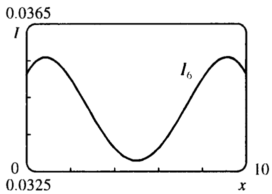
(c) Substituting $d = 10$ into the equations for $I(x)$ and $I'(x)$ gives $I_{10}(x) = \dfrac{1}{x^2 + 100} + \dfrac{1}{x^2 - 20x + 200}$ and $I'_{10}(x) = -\dfrac{2x}{(x^2 + 100)^2} - \dfrac{2\,(x - 10)}{(x^2 - 20x + 200)^2}$.

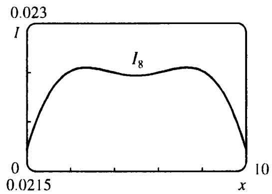


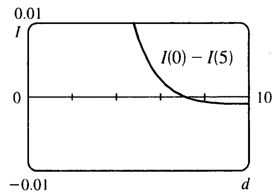


From the graphs, it seems that for $d = 10$, the intensity is minimized at the endpoints, that is, $x = 0$ and $x = 10$. The midpoint is now the most brightly lit point!

(d) From the first figures in parts (b) and (c), we see that the minimal illumination changes from the midpoint ($x = 5$ with $d = 5$) to the endpoints ($x = 0$ and $x = 10$ with $d = 10$).







So we try $d = 6$ (see the first figure) and we see that the minimum value still occurs at $x = 5$. Next, we let $d = 8$ (see the second figure) and we see that the minimum value occurs at the endpoints. It appears that for some value of d between 6 and 8, we must have minima at both the midpoint and the endpoints, that is, $I(5)$ must equal $I(0)$. To find this value of d, we solve $I(0) = I(5)$ (with $k = 1$):

$$\frac{1}{d^2} + \frac{1}{100 + d^2} = \frac{1}{25 + d^2} + \frac{1}{25 + d^2} = \frac{2}{25 + d^2} \quad \Rightarrow$$

$$(25 + d^2)(100 + d^2) + d^2(25 + d^2) = 2d^2(100 + d^2) \quad \Rightarrow$$

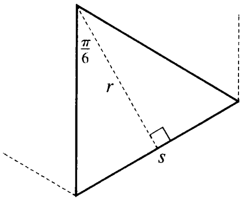
$$2500 + 125d^2 + d^4 + 25d^2 + d^4 = 200d^2 + 2d^4 \quad \Rightarrow \quad 2500 = 50d^2 \quad \Rightarrow \quad d^2 = 50 \quad \Rightarrow$$

$d = 5\sqrt{2} \approx 7.071$ (for $0 \le d \le 10$). The third figure, a graph of $I(0) - I(5)$ with d independent, confirms that $I(0) - I(5) = 0$, that is, $I(0) = I(5)$, when $d = 5\sqrt{2}$. Thus, the point of minimal illumination changes abruptly from the midpoint to the endpoints when $d = 5\sqrt{2}$.

APPLIED PROJECT The Shape of a Can

1. In this case, the amount of metal used in the making of each top or bottom is $(2r)^2 = 4r^2$. So the quantity we want to minimize is $A = 2\pi rh + 2(4r^2)$. But $V = \pi r^2 h \;\Leftrightarrow\; h = V/\pi r^2$. Substituting this expression for h in A gives $A = 2V/r + 8r^2$. Differentiating A with respect to r, we get $dA/dr = -2V/r^2 + 16r = 0 \;\Rightarrow$

$16r^3 = 2V = 2\pi r^2 h \;\Leftrightarrow\; \dfrac{h}{r} = \dfrac{8}{\pi} \approx 2.55$. This gives a minimum because $\dfrac{d^2A}{dr^2} = 16 + \dfrac{4V}{r^3} > 0$.

2. 

We need to find the area of metal used up by each end, that is, the area of each hexagon. We subdivide the hexagon into six congruent triangles, each sharing one side (s in the diagram) with the hexagon. We calculate the length of $s = 2r\tan\frac{\pi}{6} = \frac{2}{\sqrt{3}}r$, so the area of each triangle is $\frac{1}{2}sr = \frac{1}{\sqrt{3}}r^2$, and the total area of the hexagon is $6 \cdot \frac{1}{\sqrt{3}}r^2 = 2\sqrt{3}\,r^2$. So the quantity we want to minimize is $A = 2\pi rh + 2 \cdot 2\sqrt{3}\,r^2$.

Substituting for h as in Problem 1 and differentiating, we get $\dfrac{dA}{dr} = -\dfrac{2V}{r^2} + 8\sqrt{3}\,r$. Setting this equal to 0, we get

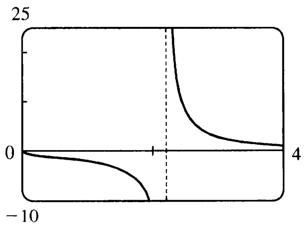
$8\sqrt{3}\,r^3 = 2V = 2\pi r^2 h \quad\Rightarrow\quad \dfrac{h}{r} = \dfrac{4\sqrt{3}}{\pi} \approx 2.21$. Again this minimizes A because $\dfrac{d^2A}{dr^2} = 8\sqrt{3} + \dfrac{4V}{r^3} > 0$.

3. Let $C = 4\sqrt{3}\,r^2 + 2\pi rh + k(4\pi r + h) = 4\sqrt{3}\,r^2 + 2\pi r\left(\dfrac{V}{\pi r^2}\right) + k\left(4\pi r + \dfrac{V}{\pi r^2}\right)$. Then

$\dfrac{dC}{dr} = 8\sqrt{3}\,r - \dfrac{2V}{r^2} + 4k\pi - \dfrac{2kV}{\pi r^3}$. Setting this equal to 0, dividing by 2 and substituting $\dfrac{V}{r^2} = \pi h$ and $\dfrac{V}{\pi r^3} = \dfrac{h}{r}$ in the second and fourth terms respectively, we get $0 = 4\sqrt{3}\,r - \pi h + 2k\pi - \dfrac{kh}{r} \quad\Leftrightarrow$

$k\left(2\pi - \dfrac{h}{r}\right) = \pi h - 4\sqrt{3}\,r \quad\Rightarrow\quad \dfrac{k}{r}\,\dfrac{2\pi - h/r}{\pi h/r - 4\sqrt{3}} = 1$. We now multiply by $\dfrac{\sqrt[3]{V}}{k}$, noting that

$\dfrac{\sqrt[3]{V}}{k}\,\dfrac{k}{r} = \sqrt[3]{\dfrac{V}{r^3}} = \sqrt[3]{\dfrac{\pi h}{r}}$, and get $\dfrac{\sqrt[3]{V}}{k} = \sqrt[3]{\dfrac{\pi h}{r}} \cdot \dfrac{2\pi - h/r}{\pi h/r - 4\sqrt{3}}$.

4. 

Let $\sqrt[3]{V}/k = T$ and $h/r = x$ so that $T(x) = \sqrt[3]{\pi x} \cdot \dfrac{2\pi - x}{\pi x - 4\sqrt{3}}$. We see from the graph of T that when the ratio $\sqrt[3]{V}/k$ is large; that is, either the volume of the can is large or the cost of joining (proportional to k) is small, the optimum value of h/r is about 2.21, but when $\sqrt[3]{V}/k$ is small, indicating small volume or expensive joining, the optimum value of h/r is larger. (The part of the graph for $\sqrt[3]{V}/k < 0$ has no physical meaning, but confirms the location of the asymptote.)

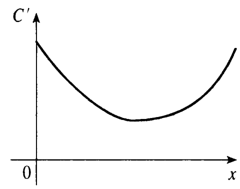
5. Our conclusion is usually true in practice. But there are exceptions, such as cans of tuna, which may have to do with the shape of a reasonable slice of tuna. And for a comfortable grip on a soda or beer can, the geometry of the human hand is a restriction on the radius. Other possible considerations are packaging, transportation and stocking constraints, aesthetic appeal and other marketing concerns. Also, there may be better models than ours which prescribe a differently shaped can in special circumstances.

4.8 Applications to Business and Economics

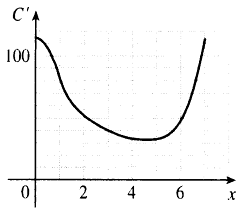
1. (a) $C(0)$ represents the fixed costs of production, such as rent, utilities, machinery etc., which are incurred even when nothing is produced.

(b) The inflection point is the point at which $C''(x)$ changes from negative to positive; that is, the marginal cost $C'(x)$ changes from decreasing to increasing. Thus, the marginal cost is minimized at the inflection point.

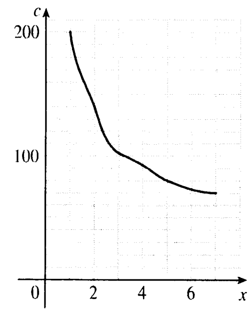
(c) The marginal cost function is $C'(x)$. We graph it as in Example 1 in Section 2.9.



2. (a) We graph C' as in Example 1 in Section 2.9.



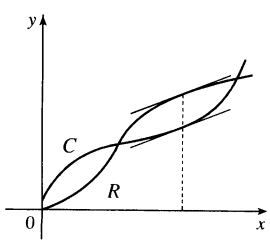
(b) By reading values of $C(x)$ from its graph, we can plot $c(x) = C(x)/x$.



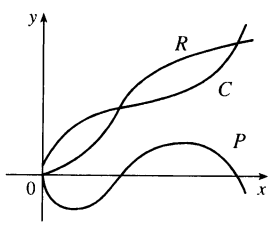
(c) Since the graph in part (b) is decreasing, we estimate that the minimum value of $c(x)$ occurs at $x = 7$. The average cost and the marginal cost are equal at that value. See the box preceding Example 1.

3. $c(x) = 21.4 - 0.002x$ and $c(x) = C(x)/x \;\Rightarrow\; C(x) = 21.4x - 0.002x^2$. $C'(x) = 21.4 - 0.004x$ and $C'(1000) = 17.4$. This means that the cost of producing the 1001st unit is about \$17.40.

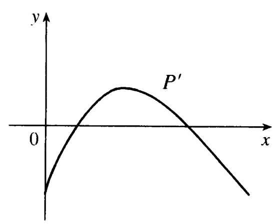
4. (a) Profit is maximized when the marginal revenue is equal to the marginal cost; that is, when R and C have equal slopes. See the box preceding Example 2.



(b) $P(x) = R(x) - C(x)$ is sketched.



(c) The marginal profit function is defined as $P'(x)$.



5. (a) The cost function is $C(x) = 40{,}000 + 300x + x^2$, so the cost at a production level of 1000 is $C(1000) = \$1{,}340{,}000$. The average cost function is $c(x) = \dfrac{C(x)}{x} = \dfrac{40{,}000}{x} + 300 + x$ and $c(1000) = \$1340/\text{unit}$. The marginal cost function is $C'(x) = 300 + 2x$ and $C'(1000) = \$2300/\text{unit}$.

(b) See the box preceding Example 1. We must have $C'(x) = c(x) \;\Leftrightarrow\; 300 + 2x = \dfrac{40{,}000}{x} + 300 + x \;\Leftrightarrow\; x = \dfrac{40{,}000}{x} \;\Rightarrow\; x^2 = 40{,}000 \;\Rightarrow\; x = \sqrt{40{,}000} = 200$. This gives a minimum value of the average cost function $c(x)$ since $c''(x) = \dfrac{80{,}000}{x^3} > 0$.

(c) The minimum average cost is $c(200) = \$700/\text{unit}$.

6. (a) $C(x) = 25{,}000 + 120x + 0.1x^2$, $C(1000) = \$245{,}000$. $c(x) = \dfrac{C(x)}{x} = \dfrac{25{,}000}{x} + 120 + 0.1x$, $c(1000) = \$245/\text{unit}$. $C'(x) = 120 + 0.2x$, $C'(1000) = \$320/\text{unit}$.

(b) We must have $C'(x) = c(x) \;\Leftrightarrow\; 120 + 0.2x = \dfrac{25{,}000}{x} + 120 + 0.1x \;\Leftrightarrow\; 0.1x = \dfrac{25{,}000}{x} \;\Rightarrow\; 0.1x^2 = 25{,}000 \;\Rightarrow\; x = \sqrt{250{,}000} = 500$. This gives a minimum since $c''(x) = \dfrac{50{,}000}{x^3} > 0$.

(c) The minimum average cost is $c(500) = \$220.00/\text{unit}$.

7. (a) $C(x) = 16{,}000 + 200x + 4x^{3/2}$, $C(1000) = 16{,}000 + 200{,}000 + 40{,}000\sqrt{10} \approx 216{,}000 + 126{,}491$, so $C(1000) \approx \$342{,}491$. $c(x) = C(x)/x = \dfrac{16{,}000}{x} + 200 + 4x^{1/2}$, $c(1000) \approx \$342.49/\text{unit}$. $C'(x) = 200 + 6x^{1/2}$, $C'(1000) = 200 + 60\sqrt{10} \approx \$389.74/\text{unit}$.

(b) We must have $C'(x) = c(x) \;\Leftrightarrow\; 200 + 6x^{1/2} = \dfrac{16{,}000}{x} + 200 + 4x^{1/2} \;\Leftrightarrow\; 2x^{3/2} = 16{,}000 \;\Leftrightarrow\; x = (8{,}000)^{2/3} = 400$ units. To check that this is a minimum, we calculate $c'(x) = \dfrac{-16{,}000}{x^2} + \dfrac{2}{\sqrt{x}} = \dfrac{2}{x^2}(x^{3/2} - 8000)$. This is negative for $x < (8000)^{2/3} = 400$, zero at $x = 400$, and positive for $x > 400$, so c is decreasing on $(0, 400)$ and increasing on $(400, \infty)$. Thus, c has an absolute minimum at $x = 400$. [*Note:* $c''(x)$ is *not* positive for all $x > 0$.]

(c) The minimum average cost is $c(400) = 40 + 200 + 80 = \$320/\text{unit}$.

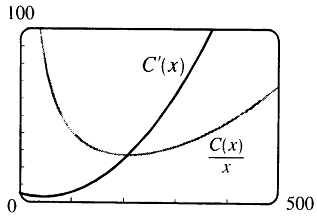
8. (a) $C(x) = 10{,}000 + 340x - 0.3x^2 + 0.0001x^3$, $C(1000) = \$150{,}000$.
$c(x) = C(x)/x = \dfrac{10{,}000}{x} + 340 - 0.3x + 0.0001x^2$, $c(1000) = \$150/\text{unit}$.
$C'(x) = 340 - 0.6x + 0.0003x^2$, $C'(1000) = \$40/\text{unit}$.

(b) We must have $C'(x) = c(x) \;\Leftrightarrow\; 340 - 0.6x + 0.0003x^2 = \dfrac{10{,}000}{x} + 340 - 0.3x + 0.0001x^2 \;\Leftrightarrow\; 0.0002x^2 = \dfrac{10{,}000}{x} + 0.3x \;\Leftrightarrow\; 0.0002x^3 - 0.3x^2 - 10{,}000 = 0 \;\Leftrightarrow\; x^3 - 1500x^2 - 50{,}000{,}000 = 0 \;\Rightarrow\; x \approx 1521.60 \approx 1522$ units. This gives a minimum since $c''(x) = \dfrac{20{,}000}{x^3} + 0.0002 > 0$.

(c) The minimum average cost is about $c(1521.60) \approx \$121.62/\text{unit}$.

9. (a) $C(x) = 3700 + 5x - 0.04x^2 + 0.0003x^3 \;\Rightarrow\; C'(x) = 5 - 0.08x + 0.0009x^2$ (marginal cost).
$c(x) = \dfrac{C(x)}{x} = \dfrac{3700}{x} + 5 - 0.04x + 0.0003x^2$ (average cost).

(b)



The graphs intersect at $(208.51, 27.45)$, so the production level that minimizes average cost is about 209 units.

(c) $c'(x) = -\dfrac{3700}{x^2} - 0.04 + 0.0006x = 0 \Rightarrow 3700 + 0.04x^2 - 0.0006x^3 = 0 \Rightarrow x_1 \approx 208.51$.

$c(x_1) \approx \$27.45/\text{unit}$.

(d) The marginal cost is given by $C'(x)$, so to find its minimum value we'll find the derivative of C'; that is, C''.

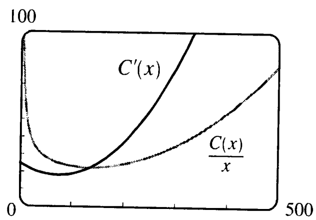
$C''(x) = -0.08 + 0.0018x = 0 \Rightarrow x_1 = \frac{800}{18} = 44.4\overline{4}$. $C'(x_1) = \$3.22/\text{unit}$.

$C'''(x) = 0.0018 > 0$ for all x, so this is the minimum marginal cost. C''' is the second derivative of C'.

10. (a) $C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3 \Rightarrow C'(x) = 25 - 0.18x + 0.0012x^2$ (marginal cost).

$c(x) = \dfrac{C(x)}{x} = \dfrac{339}{x} + 25 - 0.09x + 0.0004x^2$ (average cost).

(b)



The graphs intersect at $(135.56, 22.65)$, so the production level that minimizes average cost is about 136 units.

(c) $c'(x) = -\dfrac{339}{x^2} - 0.09 + 0.0008x = 0 \Rightarrow x_1 \approx 135.56$. $c(x_1) \approx \$22.65/\text{unit}$.

(d) $C''(x) = -0.18 + 0.0024x = 0 \Rightarrow x = \frac{1800}{24} = 75$. $C'(75) = \$18.25/\text{unit}$.
$C'''(x) = 0.0024 > 0$ for all x, so this is the minimum marginal cost.

11. $C(x) = 680 + 4x + 0.01x^2$, $p(x) = 12 \Rightarrow R(x) = xp(x) = 12x$. If the profit is maximum, then $R'(x) = C'(x) \Rightarrow 12 = 4 + 0.02x \Rightarrow 0.02x = 8 \Rightarrow x = 400$. The profit is maximized if $P''(x) < 0$, but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now $R''(x) = 0 < 0.02 = C''(x)$, so $x = 400$ gives a maximum.

12. $C(x) = 680 + 4x + 0.01x^2$, $p(x) = 12 - x/500$. Then $R(x) = xp(x) = 12x - x^2/500$. If the profit is maximum, then $R'(x) = C'(x)$ [See the box preceding Example 2.] $\Leftrightarrow 12 - x/250 = 4 + 0.02x \Leftrightarrow 8 = 0.024x \Leftrightarrow x = 8/0.024 = \frac{1000}{3}$. The profit is maximized if $P''(x) < 0$, but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now $R''(x) = -\frac{1}{250} < 0.02 = C''(x)$, so $x = \frac{1000}{3}$ gives a maximum.

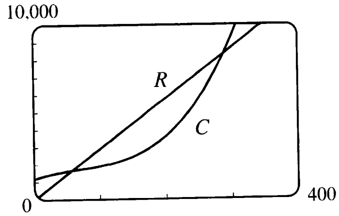
13. $C(x) = 1450 + 36x - x^2 + 0.001x^3$, $p(x) = 60 - 0.01x$. Then $R(x) = xp(x) = 60x - 0.01x^2$. If the profit is maximum, then $R'(x) = C'(x) \Leftrightarrow 60 - 0.02x = 36 - 2x + 0.003x^2 \Rightarrow 0.003x^2 - 1.98x - 24 = 0$. By the quadratic formula, $x = \dfrac{1.98 \pm \sqrt{(-1.98)^2 + 4(0.003)(24)}}{2(0.003)} = \dfrac{1.98 \pm \sqrt{4.2084}}{0.006}$. Since $x > 0$, $x \approx (1.98 + 2.05)/0.006 \approx 672$. Now $R''(x) = -0.02$ and $C''(x) = -2 + 0.006x \Rightarrow C''(672) = 2.032$ $\Rightarrow R''(672) < C''(672) \Rightarrow$ there is a maximum at $x = 672$.

14. $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$, $p(x) = 1700 - 7x$. Then $R(x) = xp(x) = 1700x - 7x^2$. If the profit is maximum, then $R'(x) = C'(x) \Leftrightarrow 1700 - 14x = 500 - 3.2x + 0.012x^2 \Leftrightarrow$ $0.012x^2 + 10.8x - 1200 = 0 \Leftrightarrow x^2 + 900x - 100{,}000 = 0 \Leftrightarrow (x + 1000)(x - 100) = 0 \Leftrightarrow x = 100$ (since $x > 0$). The profit is maximized if $P''(x) < 0$, but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now $R''(x) = -14 < -3.2 + 0.024x = C''(x)$ for $x > 0$, so there is a maximum at $x = 100$.

15. $C(x) = 0.001x^3 - 0.3x^2 + 6x + 900$. The marginal cost is $C'(x) = 0.003x^2 - 0.6x + 6$. $C'(x)$ is increasing when $C''(x) > 0 \Leftrightarrow 0.006x - 0.6 > 0 \Leftrightarrow x > 0.6/0.006 = 100$. So $C'(x)$ starts to increase when $x = 100$.

16. $C(x) = 0.0002x^3 - 0.25x^2 + 4x + 1500$. The marginal cost is $C'(x) = 0.0006x^2 - 0.50x + 4$. $C'(x)$ is increasing when $C''(x) > 0 \Leftrightarrow 0.0012x - 0.5 > 0 \Leftrightarrow x > 0.5/0.0012 \approx 417$. So $C'(x)$ starts to increase when $x = 417$.

17. (a) $C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3$.
$R(x) = xp(x) = 29x - 0.00021x^2$.
Since the profit is maximized when $R'(x) = C'(x)$, we examine the curves R and C in the figure, looking for x-values at which the slopes of the tangent lines are equal. It appears that $x = 200$ is a good estimate.



(b) $R'(x) = C'(x) \Rightarrow 29 - 0.00042x = 12 - 0.2x + 0.0015x^2 \Rightarrow 0.0015x^2 - 0.19958x - 17 = 0 \Rightarrow$ $x \approx 192.06$ (for $x > 0$). As in Exercise 11, $R''(x) < C''(x) \Rightarrow -0.00042 < -0.2 + 0.003x \Leftrightarrow$ $0.003x > 0.19958 \Leftrightarrow x > 66.5$. Our value of 192 is in this range, so we have a maximum profit when we produce 192 yards of fabric.

18. (a) Cost = setup cost + manufacturing cost $\Rightarrow C(x) = 500 + m(x) = 500 + 20x - 5x^{3/4} + 0.01x^2$.
We can solve $x(p) = 320 - 7.7p$ for p in terms of x to find the demand (or price) function.
$x = 320 - 7.7p \Rightarrow 7.7p = 320 - x \Rightarrow p(x) = \dfrac{320 - x}{7.7}$. $R(x) = xp(x) = \dfrac{320x - x^2}{7.7}$.

(b) $C'(x) = R'(x) \Rightarrow 20 - \frac{15}{4}x^{-1/4} + 0.02x = \dfrac{320 - 2x}{7.7} \Rightarrow x \approx 81.53$ planes, and $p(x) = \$30.97$ million. The maximum profit associated with these values is about \$463.59 million.

19. (a) We are given that the demand function p is linear and $p(27{,}000) = 10$, $p(33{,}000) = 8$, so the slope is $\frac{10 - 8}{27{,}000 - 33{,}000} = -\frac{1}{3000}$ and an equation of the line is $y - 10 = \left(-\frac{1}{3000}\right)(x - 27{,}000) \Rightarrow$ $y = p(x) = -\frac{1}{3000}x + 19 = 19 - (x/3000)$.

(b) The revenue is $R(x) = xp(x) = 19x - (x^2/3000) \Rightarrow R'(x) = 19 - (x/1500) = 0$ when $x = 28{,}500$. Since $R''(x) = -1/1500 < 0$, the maximum revenue occurs when $x = 28{,}500 \Rightarrow$ the price is $p(28{,}500) = \$9.50$.

20. (a) Let $p(x)$ be the demand function. Then $p(x)$ is linear and $y = p(x)$ passes through $(20, 10)$ and $(18, 11)$, so the slope is $-\frac{1}{2}$ and an equation of the line is $y - 10 = -\frac{1}{2}(x - 20) \iff y = -\frac{1}{2}x + 20$. Thus, the demand is $p(x) = -\frac{1}{2}x + 20$ and the revenue is $R(x) = xp(x) = -\frac{1}{2}x^2 + 20x$.

(b) The cost is $C(x) = 6x$, so the profit is $P(x) = R(x) - C(x) = -\frac{1}{2}x^2 + 14x$. Then $0 = P'(x) = -x + 14$ $\Rightarrow$ $x = 14$. Since $P''(x) = -1 < 0$, the selling price for maximum profit is $p(14) = -\frac{1}{2}(14) + 20 = \$13$.

21. (a) As in Example 3, we see that the demand function p is linear. We are given that $p(1000) = 450$ and deduce that $p(1100) = 440$, since a \$10 reduction in price increases sales by 100 per week. The slope for p is $\frac{440 - 450}{1100 - 1000} = -\frac{1}{10}$, so an equation is $p - 450 = -\frac{1}{10}(x - 1000)$ or $p(x) = -\frac{1}{10}x + 550$.

(b) $R(x) = xp(x) = -\frac{1}{10}x^2 + 550x$. $R'(x) = -\frac{1}{5}x + 550 = 0$ when $x = 5(550) = 2750$. $p(2750) = 275$, so the rebate should be $450 - 275 = \$175$.

(c) $C(x) = 68{,}000 + 150x$ $\Rightarrow$

$P(x) = R(x) - C(x) = -\frac{1}{10}x^2 + 550x - 68{,}000 - 150x = -\frac{1}{10}x^2 + 400x - 68{,}000$,

$P'(x) = -\frac{1}{5}x + 400 = 0$ when $x = 2000$. $p(2000) = 350$. Therefore, the rebate to maximize profits should be $450 - 350 = \$100$.

22. Let x denote the number of \$10 increases in rent. Then the price is $p(x) = 800 + 10x$, and the number of units occupied is $100 - x$. Now the revenue is

$$\begin{aligned} R(x) &= (\text{rental price per unit}) \times (\text{number of units rented}) \\ &= (800 + 10x)(100 - x) = -10x^2 + 200x + 80{,}000 \text{ for } 0 \le x \le 100 \quad \Rightarrow \end{aligned}$$

$R'(x) = -20x + 200 = 0 \iff x = 10$. This is a maximum since $R''(x) = -20 < 0$ for all x. Now we must check the value of $R(x) = (800 + 10x)(100 - x)$ at $x = 10$ and at the endpoints of the domain to see which value of x gives the maximum value of R. $R(0) = 80{,}000$, $R(10) = (900)(90) = 81{,}000$, and $R(100) = (1800)(0) = 0$. Thus, the maximum revenue of \$81,000/week occurs when 90 units are occupied at a rent of \$900/week.

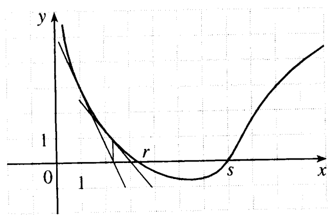
23. If the reorder quantity is x, then the manager places $\dfrac{800}{x}$ orders per year. Storage costs for the year are $\frac{1}{2}x \cdot 4 = 2x$ dollars. Handling costs are \$100 per delivery, for a total of $\dfrac{800}{x} \cdot 100 = \dfrac{80{,}000}{x}$ dollars. The total costs for the year are $C(x) = 2x + \dfrac{80{,}000}{x}$. To minimize $C(x)$, we calculate

$C'(x) = 2 - \dfrac{80{,}000}{x^2} = \dfrac{2}{x^2}(x^2 - 40{,}000)$. This is negative when $x < 200$, zero when $x = 200$, and positive when $x > 200$, so C is decreasing on $(0, 200)$ and increasing on $(200, \infty)$, reaching its absolute minimum when $x = 200$. Thus, the optimal reorder quantity is 200 cases. The manager will place 4 orders per year for a total cost of $C(200) = \$800$.

24. She will have A/n dollars after each withdrawal and 0 dollars just before the next withdrawal, so her average cash balance at any given time is $\frac{1}{2}(A/n + 0) = A/(2n)$. The transaction costs for n withdrawals are nT. The lost interest cost on the average cash balance is $[A/(2n)] \cdot R$. Thus, the total cost for n transactions is $C(n) = nT + \dfrac{AR}{2n}$. Now $C'(n) = T - \dfrac{AR}{2n^2}$ and $C'(n) = 0 \;\Rightarrow\; \dfrac{AR}{2n^2} = T \;\Rightarrow\; n^2 = \dfrac{AR}{2T} \;\Rightarrow$ $n = \sqrt{\dfrac{AR}{2T}}$, the value of n that minimizes total costs since $C''(n) = -\dfrac{AR}{n^3} < 0$. Thus, the optimal average cash balance is $\dfrac{A}{2n} = \dfrac{A\sqrt{2T}}{2\sqrt{AR}} = \dfrac{\sqrt{AT}}{\sqrt{2R}} = \sqrt{\dfrac{AT}{2R}}$.

4.9 Newton's Method

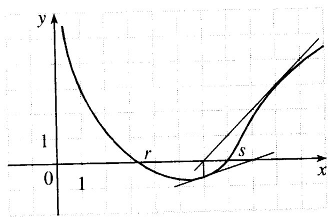
1. (a)



The tangent line at $x = 1$ intersects the x-axis at $x \approx 2.3$, so $x_2 \approx 2.3$. The tangent line at $x = 2.3$ intersects the x-axis at $x \approx 3$, so $x_3 \approx 3.0$.

(b) $x_1 = 5$ would *not* be a better first approximation than $x_1 = 1$ since the tangent line is nearly horizontal. In fact, the second approximation for $x_1 = 5$ appears to be to the left of $x = 1$.

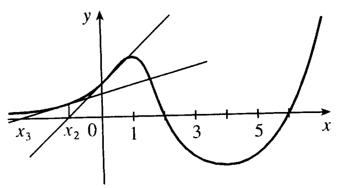
2.



The tangent line at $x = 9$ intersects the x-axis at $x \approx 6.0$, so $x_2 \approx 6.0$. The tangent line at $x = 6.0$ intersects the x-axis at $x \approx 8.0$, so $x_3 \approx 8.0$.

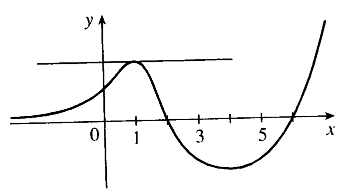
3. Since $x_1 = 3$ and $y = 5x - 4$ is tangent to $y = f(x)$ at $x = 3$, we simply need to find where the tangent line intersects the x-axis. $y = 0 \;\Rightarrow\; 5x_2 - 4 = 0 \;\Rightarrow\; x_2 = \frac{4}{5}$.

4. (a)



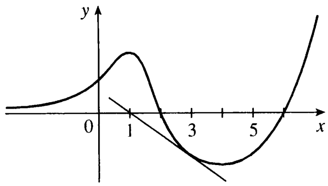
If $x_1 = 0$, then x_2 is negative, and x_3 is even more negative. The sequence of approximations does not converge, that is, Newton's method fails.

(b)



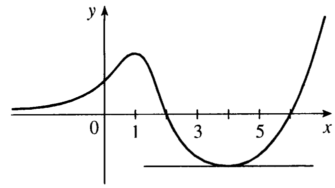
If $x_1 = 1$, the tangent line is horizontal and Newton's method fails.

(c)



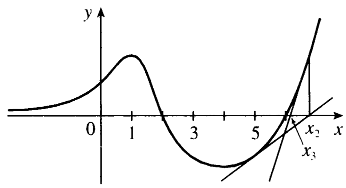
If $x_1 = 3$, then $x_2 = 1$ and we have the same situation as in part (b). Newton's method fails again.

(d)



If $x_1 = 4$, the tangent line is horizontal and Newton's method fails.

(e)



If $x_1 = 5$, then x_2 is greater than 6, x_3 gets closer to 6, and the sequence of approximations converges to 6. Newton's method succeeds!

5. $f(x) = x^3 + 2x - 4 \;\Rightarrow\; f'(x) = 3x^2 + 2$, so $x_{n+1} = x_n - \dfrac{x_n^3 + 2x_n - 4}{3x_n^2 + 2}$. Now $x_1 = 1 \;\Rightarrow$

$$x_2 = 1 - \frac{1 + 2 - 4}{3 \cdot 1^2 + 2} = 1 - \frac{-1}{5} = 1.2 \;\Rightarrow\; x_3 = 1.2 - \frac{(1.2)^3 + 2(1.2) - 4}{3(1.2)^2 + 2} \approx 1.1797.$$

6. $f(x) = x^3 - x^2 - 1 \;\Rightarrow\; f'(x) = 3x^2 - 2x$, so $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^3 - x_n^2 - 1}{3x_n^2 - 2x_n}$.

Now $x_1 = 1 \;\Rightarrow\; x_2 = 1 - \dfrac{1 - 1 - 1}{3 - 2} = 2 \;\Rightarrow\; x_3 = 2 - \dfrac{2^3 - 2^2 - 1}{3 \cdot 2^2 - 2 \cdot 2} = 1.625.$

7. $f(x) = x^4 - 20 \;\Rightarrow\; f'(x) = 4x^3$, so $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^4 - 20}{4x_n^3}$.

Now $x_1 = 2 \;\Rightarrow\; x_2 = 2 - \dfrac{2^4 - 20}{4(2)^3} = 2.125 \;\Rightarrow\; x_3 = 2.125 - \dfrac{(2.125)^4 - 20}{4(2.125)^3} \approx 2.1148.$

8. $f(x) = x^5 + 2 \;\Rightarrow\; f'(x) = 5x^4$, so $x_{n+1} = x_n - \dfrac{x_n^5 + 2}{5x_n^4}$. Now $x_1 = -1 \;\Rightarrow$

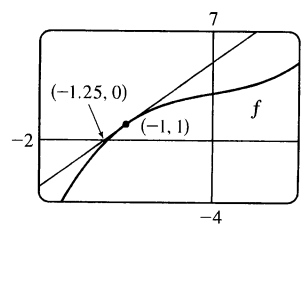
$$x_2 = -1 - \frac{(-1)^5 + 2}{5 \cdot (-1)^4} = -1 - \frac{1}{5} = -1.2 \;\Rightarrow\; x_3 = -1.2 - \frac{(-1.2)^5 + 2}{5(-1.2)^4} \approx -1.1529.$$

9. $f(x) = x^3 + x + 3 \;\Rightarrow\; f'(x) = 3x^2 + 1$, so

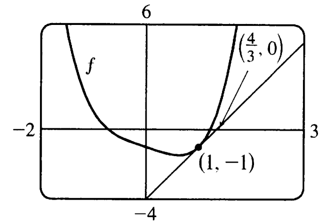
$x_{n+1} = x_n - \dfrac{x_n^3 + x_n + 3}{3x_n^2 + 1}$. Now $x_1 = -1 \;\Rightarrow$

$$x_2 = -1 - \frac{(-1)^3 + (-1) + 3}{3(-1)^2 + 1} = -1 - \frac{-1 - 1 + 3}{3 + 1} = -1 - \frac{1}{4} = -1.25.$$

Newton's method follows the tangent line at $(-1, 1)$ down to its intersection with the x-axis at $(-1.25, 0)$, giving the second approximation $x_2 = -1.25$.

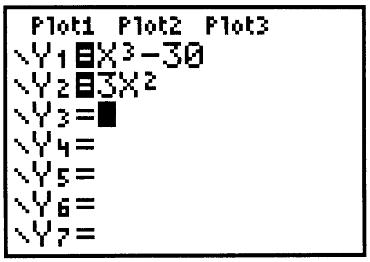


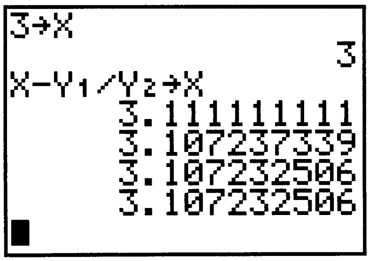
10. $f(x) = x^4 - x - 1 \quad\Rightarrow\quad f'(x) = 4x^3 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^4 - x_n - 1}{4x_n^3 - 1}$. Now $x_1 = 1 \quad\Rightarrow$ $x_2 = 1 - \dfrac{1^4 - 1 - 1}{4 \cdot 1^3 - 1} = 1 - \dfrac{-1}{3} = \dfrac{4}{3}$. Newton's method follows the tangent line at $(1, -1)$ up to its intersection with the x-axis at $\left(\frac{4}{3}, 0\right)$, giving the second approximation $x_2 = \frac{4}{3}$.



11. To approximate $x = \sqrt[3]{30}$ (so that $x^3 = 30$), we can take $f(x) = x^3 - 30$. So $f'(x) = 3x^2$, and thus, $x_{n+1} = x_n - \dfrac{x_n^3 - 30}{3x_n^2}$. Since $\sqrt[3]{27} = 3$ and 27 is close to 30, we'll use $x_1 = 3$. We need to find approximations until they agree to eight decimal places. $x_1 = 3 \quad\Rightarrow\quad x_2 \approx 3.11111111$, $x_3 \approx 3.10723734$, $x_4 \approx 3.10723251 \approx x_5$. So $\sqrt[3]{30} \approx 3.10723251$, to eight decimal places.

Here is a quick and easy method for finding the iterations for Newton's method on a programmable calculator. (The screens shown are from the TI-83 Plus, but the method is similar on other calculators.) Assign $f(x) = x^3 - 30$ to Y_1, and $f'(x) = 3x^2$ to Y_2. Now store $x_1 = 3$ in X and then enter $\mathrm{X} - \mathrm{Y}_1/\mathrm{Y}_2 \to \mathrm{X}$ to get $x_2 = 3.\overline{1}$. By successively pressing the ENTER key, you get the approximations $x_3, x_4, \ldots$.





In Derive, load the utility file `SOLVE`. Enter `NEWTON(x^3-30,x,3)` and then `APPROXIMATE` to get [3, 3.11111111, 3.10723733, 3.10723250, 3.10723250]. You can request a specific iteration by adding a fourth argument. For example, `NEWTON(x^3-30,x,3,2)` gives [3, 3.11111111, 3.10723733].

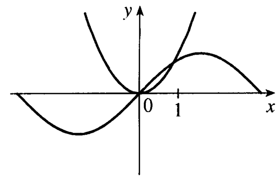
In Maple, make the assignments $f := x \to x\hat{\ }3 - 30;$, $g := x \to x - f(x)/D(f)(x);$, and $x := 3.;$. Repeatedly execute the command $x := g(x);$ to generate successive approximations.

In Mathematica, make the assignments $f[x\_] := x\hat{\ }3 - 30$, $g[x\_] := x - f[x]/f'[x]$, and $x = 3$. Repeatedly execute the command $x = g[x]$ to generate successive approximations.

12. $f(x) = x^7 - 1000 \quad\Rightarrow\quad f'(x) = 7x^6$, so $x_{n+1} = x_n - \dfrac{x_n^7 - 1000}{7x_n^6}$. We need to find approximations until they agree to eight decimal places. $x_1 = 3 \quad\Rightarrow\quad x_2 \approx 2.76739173$, $x_3 \approx 2.69008741$, $x_4 \approx 2.68275645$, $x_5 \approx 2.68269580 \approx x_6$. So $\sqrt[7]{1000} \approx 2.68269580$, to eight decimal places.

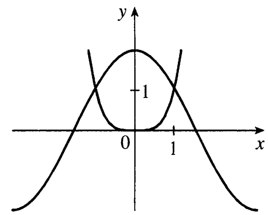
13. $f(x) = 2x^3 - 6x^2 + 3x + 1 \quad\Rightarrow\quad f'(x) = 6x^2 - 12x + 3 \quad\Rightarrow\quad x_{n+1} = x_n - \dfrac{2x_n^3 - 6x_n^2 + 3x_n + 1}{6x_n^2 - 12x_n + 3}$. We need to find approximations until they agree to six decimal places. $x_1 = 2.5 \quad\Rightarrow\quad x_2 \approx 2.285714$, $x_3 \approx 2.228824$, $x_4 \approx 2.224765$, $x_5 \approx 2.224745 \approx x_6$. So the root is 2.224745, to six decimal places.

14. $f(x) = x^4 + x - 4 \quad\Rightarrow\quad f'(x) = 4x^3 + 1 \quad\Rightarrow\quad x_{n+1} = x_n - \dfrac{x_n^4 + x_n - 4}{4x_n^3 + 1}$. $x_1 = 1.5 \quad\Rightarrow\quad x_2 \approx 1.323276$, $x_3 \approx 1.285346$, $x_4 \approx 1.283784$, $x_5 \approx 1.283782 \approx x_6$. So the root is 1.283782, to six decimal places.



15. $\sin x = x^2$, so $f(x) = \sin x - x^2 \quad\Rightarrow\quad f'(x) = \cos x - 2x \quad\Rightarrow$

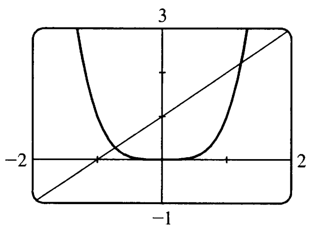
$x_{n+1} = x_n - \dfrac{\sin x_n - x_n^2}{\cos x_n - 2x_n}$. From the figure, the positive root of $\sin x = x^2$ is near 1. $x_1 = 1 \quad\Rightarrow\quad x_2 \approx 0.891396$, $x_3 \approx 0.876985$, $x_4 \approx 0.876726 \approx x_5$. So the positive root is 0.876726, to six decimal places.



16. $2\cos x = x^4$, so $f(x) = 2\cos x - x^4 \quad\Rightarrow\quad f'(x) = -2\sin x - 4x^3$

$\Rightarrow\quad x_{n+1} = x_n - \dfrac{2\cos x_n - x_n^4}{-2\sin x_n - 4x_n^3}$. From the figure, the positive root of $2\cos x = x^4$ is near 1. $x_1 = 1 \quad\Rightarrow\quad x_2 \approx 1.014184$, $x_3 \approx 1.013958 \approx x_4$. So the positive root is 1.013958, to six decimal places.

17.

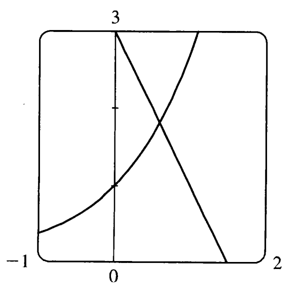


From the graph, we see that there appear to be points of intersection near $x = -0.7$ and $x = 1.2$. Solving $x^4 = 1 + x$ is the same as solving $f(x) = x^4 - x - 1 = 0$. $f(x) = x^4 - x - 1 \quad\Rightarrow\quad f'(x) = 4x^3 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^4 - x_n - 1}{4x_n^3 - 1}$.

$x_1 = -0.7$	$x_1 = 1.2$
$x_2 \approx -0.725253$	$x_2 \approx 1.221380$
$x_3 \approx -0.724493$	$x_3 \approx 1.220745$
$x_4 \approx -0.724492 \approx x_5$	$x_4 \approx 1.220744 \approx x_5$

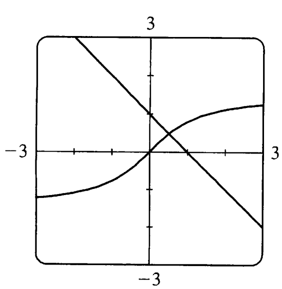
To six decimal places, the roots of the equation are -0.724492 and 1.220744.

18.



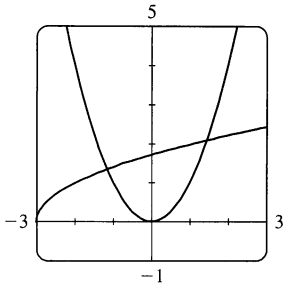
From the graph, there appears to be a point of intersection near $x = 0.6$. Solving $e^x = 3 - 2x$ is the same as solving $f(x) = e^x + 2x - 3 = 0$. $f(x) = e^x + 2x - 3 \quad\Rightarrow\quad f'(x) = e^x + 2$, so $x_{n+1} = x_n - \dfrac{e^{x_n} + 2x_n - 3}{e^{x_n} + 2}$. Now $x_1 = 0.6 \quad\Rightarrow\quad x_2 \approx 0.594213$, $x_3 \approx 0.594205 \approx x_4$. So to six decimal places, the root of the equation is 0.594205.

19.



From the graph, there appears to be a point of intersection near $x = 0.5$. Solving $\tan^{-1} x = 1 - x$ is the same as solving $f(x) = \tan^{-1} x + x - 1 = 0$. $f(x) = \tan^{-1} x + x - 1 \quad\Rightarrow$ $f'(x) = \dfrac{1}{1+x^2} + 1$, so $x_{n+1} = x_n - \dfrac{\tan^{-1} x_n + x_n - 1}{1/(1+x_n^2) + 1}$. $x_1 = 0.5 \quad\Rightarrow\quad x_2 \approx 0.520196$, $x_3 \approx 0.520269 \approx x_4$. So to six decimal places, the root of the equation is 0.520269.

20.



From the graph, we see that there appear to be points of intersection near $x = -1.2$ and $x = 1.5$. Solving $\sqrt{x+3} = x^2$ is the same as solving $f(x) = x^2 - \sqrt{x+3} = 0$. $f(x) = x^2 - \sqrt{x+3} \;\Rightarrow$

$f'(x) = 2x - \dfrac{1}{2\sqrt{x+3}}$, so $x_{n+1} = x_n - \dfrac{x_n^2 - \sqrt{x_n+3}}{2x_n - 1\big/\left(2\sqrt{x_n+3}\right)}$.

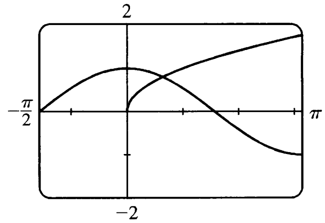
$x_1 = -1.2$	$x_1 = 1.5$
$x_2 \approx -1.164526$	$x_2 \approx 1.453449$
$x_3 \approx -1.164035 \approx x_4$	$x_3 \approx 1.452627 \approx x_4$

To six decimal places, the roots of the equation are -1.164035 and 1.452627.

21. From the graph, there appears to be a point of intersection near $x = 0.6$. Solving $\cos x = \sqrt{x}$ is the same as solving $f(x) = \cos x - \sqrt{x} = 0$. $f(x) = \cos x - \sqrt{x} \;\Rightarrow\; f'(x) = -\sin x - 1/(2\sqrt{x})$, so

$x_{n+1} = x_n - \dfrac{\cos x_n - \sqrt{x_n}}{-\sin x_n - 1/(2\sqrt{x})}$. Now $x_1 = 0.6 \;\Rightarrow$

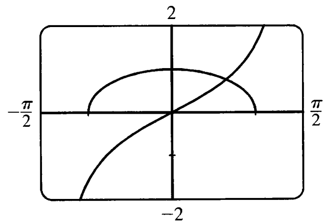
$x_2 \approx 0.641928$, $x_3 \approx 0.641714 \approx x_4$. To six decimal places, the root of the equation is 0.641714.



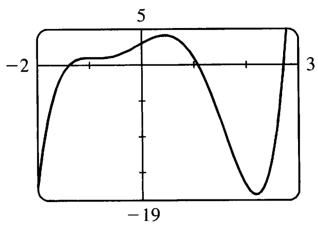
22. From the graph, there appears to be a point of intersection near $x = 0.7$. Solving $\tan x = \sqrt{1-x^2}$ is the same as solving $f(x) = \tan x - \sqrt{1-x^2} = 0$. $f(x) = \tan x - \sqrt{1-x^2} \;\Rightarrow$

$f'(x) = \sec^2 x + x/\sqrt{1-x^2}$, so $x_{n+1} = x_n - \dfrac{\tan x_n - \sqrt{1-x_n^2}}{\sec^2 x_n + x_n/\sqrt{1-x_n^2}}$.

$x_1 = 0.7 \;\Rightarrow\; x_2 \approx 0.652356$, $x_3 \approx 0.649895$, $x_4 \approx 0.649889 \approx x_5$. To six decimal places, the root of the equation is 0.649889.



23.



$f(x) = x^5 - x^4 - 5x^3 - x^2 + 4x + 3 \;\Rightarrow$

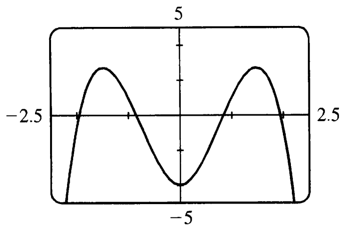
$f'(x) = 5x^4 - 4x^3 - 15x^2 - 2x + 4 \;\Rightarrow$

$x_{n+1} = x_n - \dfrac{x_n^5 - x_n^4 - 5x_n^3 - x_n^2 + 4x_n + 3}{5x_n^4 - 4x_n^3 - 15x_n^2 - 2x_n + 4}$. From the graph of f, there appear to be roots near -1.4, 1.1, and 2.7.

$x_1 = -1.4$	$x_1 = 1.1$	$x_1 = 2.7$
$x_2 \approx -1.39210970$	$x_2 \approx 1.07780402$	$x_2 \approx 2.72046250$
$x_3 \approx -1.39194698$	$x_3 \approx 1.07739442$	$x_3 \approx 2.71987870$
$x_4 \approx -1.39194691 \approx x_5$	$x_4 \approx 1.07739428 \approx x_5$	$x_4 \approx 2.71987822 \approx x_5$

To eight decimal places, the roots of the equation are -1.39194691, 1.07739428, and 2.71987822.

24.

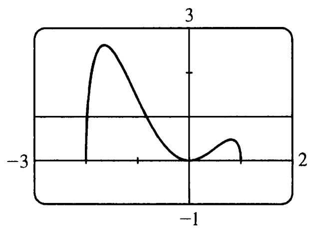


Solving $x^2(4-x^2)=\dfrac{4}{x^2+1}$ is the same as solving $f(x)=4x^2-x^4-\dfrac{4}{x^2+1}=0$. $f'(x)=8x-4x^3+\dfrac{8x}{(x^2+1)^2}$ $\Rightarrow$ $x_{n+1}=x_n-\dfrac{4x_n^2-x_n^4-4/(x_n^2+1)}{8x_n-4x_n^3+8x_n/(x_n^2+1)^2}$. From the graph of $f(x)$, there appear to be roots near $x=\pm1.9$ and $x=\pm0.8$. Since f is even, we only need to find the positive roots.

$x_1=0.8$	$x_1=1.9$
$x_2\approx0.84287645$	$x_2\approx1.94689103$
$x_3\approx0.84310820$	$x_3\approx1.94383891$
$x_4\approx0.84310821\approx x_5$	$x_4\approx1.94382538\approx x_5$

To eight decimal places, the roots of the equation are ±0.84310821 and ±1.94382538.

25.



From the graph, $y=x^2\sqrt{2-x-x^2}$ and $y=1$ intersect twice, at $x\approx-2$ and at $x\approx-1$. $f(x)=x^2\sqrt{2-x-x^2}-1$ $\Rightarrow$

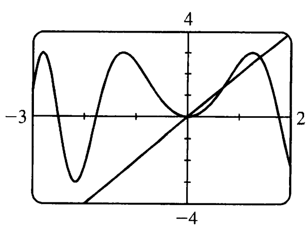
$$\begin{aligned}f'(x)&=x^2\cdot\tfrac{1}{2}(2-x-x^2)^{-1/2}(-1-2x)+(2-x-x^2)^{1/2}\cdot2x\\&=\tfrac{1}{2}x(2-x-x^2)^{-1/2}\left[x(-1-2x)+4(2-x-x^2)\right]\\&=\frac{x(8-5x-6x^2)}{2\sqrt{(2+x)(1-x)}},\end{aligned}$$

so $x_{n+1}=x_n-\dfrac{x_n^2\sqrt{2-x_n-x_n^2}-1}{\dfrac{x_n(8-5x_n-6x_n^2)}{2\sqrt{(2+x_n)(1-x_n)}}}$. Trying $x_1=-2$ won't work because $f'(-2)$ is undefined, so we'll try $x_1=-1.95$.

$x_1=-1.95$	$x_1=-0.8$
$x_2\approx-1.98580357$	$x_2\approx-0.82674444$
$x_3\approx-1.97899778$	$x_3\approx-0.82646236$
$x_4\approx-1.97807848$	$x_4\approx-0.82646233\approx x_5$
$x_5\approx-1.97806682$	
$x_6\approx-1.97806681\approx x_7$	

To eight decimal places, the roots of the equation are -1.97806681 and -0.82646233.

26.

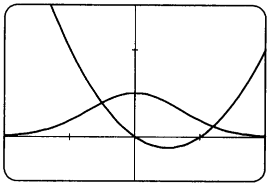


From the equations $y=3\sin(x^2)$ and $y=2x$ and the graph, we deduce that one root of the equation $3\sin(x^2)=2x$ is $x=0$. We also see that the graphs intersect at approximately $x=0.7$ and $x=1.4$. $f(x)=3\sin(x^2)-2x$ $\Rightarrow$ $f'(x)=3\cos(x^2)\cdot2x-2$, so $x_{n+1}=x_n-\dfrac{3\sin(x_n^2)-2x_n}{6x_n\cos(x_n^2)-2}$.

$x_1=0.7$	$x_1=1.4$
$x_2\approx0.69303689$	$x_2\approx1.39530295$
$x_3\approx0.69299996\approx x_4$	$x_3\approx1.39525078$
	$x_4\approx1.39525077\approx x_5$

To eight decimal places, the roots of the equation are 0.69299996 and 1.39525077.

27.



From the graph, we see that $y = e^{-x^2}$ and $y = x^2 - x$ intersect twice. Good first approximations are $x = -0.5$ and $x = 1.1$.

$f(x) = e^{-x^2} - x^2 + x \;\Rightarrow\; f'(x) = -2xe^{-x^2} - 2x + 1$, so

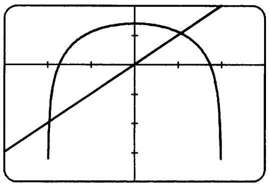
$$x_{n+1} = x_n - \frac{e^{-x_n^2} - x_n^2 + x_n}{-2x_n e^{-x_n^2} - 2x_n + 1}.$$

$x_1 = -0.5$
$x_2 \approx -0.51036446$
$x_3 \approx -0.51031156 \approx x_4$

$x_1 = 1.1$
$x_2 \approx 1.20139754$
$x_3 \approx 1.19844118$
$x_4 \approx 1.19843871 \approx x_5$

To eight decimal places, the roots of the equation are -0.51031156 and 1.19843871.

28.



From the graph, $y = \ln(4 - x^2)$ and $y = x$ intersect twice, at $x \approx -2$ and at $x \approx 1$. $f(x) = \ln(4 - x^2) - x \;\Rightarrow$

$f'(x) = \dfrac{-2x}{4 - x^2} - 1$, so $x_{n+1} = x_n - \dfrac{\ln(4 - x_n^2) - x_n}{[-2x_n/(4 - x_n^2)] - 1}$.

Trying $x_1 = -2$ won't work because it's not in the domain of $y = \ln(4 - x^2)$. Trying $x_1 = -1.9$ also fails after one iteration because the approximation x_2 is less than -2. We try $x_1 = -1.99$.

$x_1 = -1.99$
$x_2 \approx -1.97753026$
$x_3 \approx -1.96741777$
$x_4 \approx -1.96475281$
$x_5 \approx -1.96463580$
$x_6 \approx -1.96463560 \approx x_7$

$x_1 = 1.1$
$x_2 \approx 1.05864851$
$x_3 \approx 1.05800655$
$x_4 \approx 1.05800640 \approx x_5$

To eight decimal places, the roots of the equation are -1.96463560 and 1.05800640.

29. (a) $f(x) = x^2 - a \;\Rightarrow\; f'(x) = 2x$, so Newton's method gives

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = x_n - \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

(b) Using (a) with $a = 1000$ and $x_1 = \sqrt{900} = 30$, we get $x_2 \approx 31.666667$, $x_3 \approx 31.622807$, and $x_4 \approx 31.622777 \approx x_5$. So $\sqrt{1000} \approx 31.622777$.

30. (a) $f(x) = \dfrac{1}{x} - a \;\Rightarrow\; f'(x) = -\dfrac{1}{x^2}$, so $x_{n+1} = x_n - \dfrac{1/x_n - a}{-1/x_n^2} = x_n + x_n - ax_n^2 = 2x_n - ax_n^2$.

(b) Using (a) with $a = 1.6894$ and $x_1 = \frac{1}{2} = 0.5$, we get $x_2 = 0.5754$, $x_3 \approx 0.588485$, and $x_4 \approx 0.588789 \approx x_5$. So $1/1.6984 \approx 0.588789$.

31. $f(x) = x^3 - 3x + 6 \;\Rightarrow\; f'(x) = 3x^2 - 3$. If $x_1 = 1$, then $f'(x_1) = 0$ and the tangent line used for approximating x_2 is horizontal. Attempting to find x_2 results in trying to divide by zero.

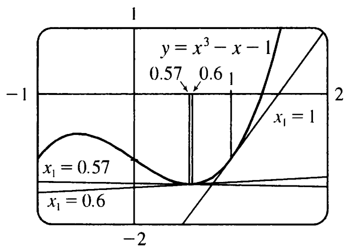
32. $x^3 - x = 1 \;\Leftrightarrow\; x^3 - x - 1 = 0$. $f(x) = x^3 - x - 1 \;\Rightarrow\; f'(x) = 3x^2 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^3 - x_n - 1}{3x_n^2 - 1}$.

(a) $x_1 = 1$, $x_2 = 1.5$, $x_3 \approx 1.347826$, $x_4 \approx 1.325200$, $x_5 \approx 1.324718 \approx x_6$

(b) $x_1 = 0.6$, $x_2 = 17.9$, $x_3 \approx 11.946802$, $x_4 \approx 7.985520$, $x_5 \approx 5.356909$, $x_6 \approx 3.624996$, $x_7 \approx 2.505589$, $x_8 \approx 1.820129$, $x_9 \approx 1.461044$, $x_{10} \approx 1.339323$, $x_{11} \approx 1.324913$, $x_{12} \approx 1.324718 \approx x_{13}$

(c) $x_1 = 0.57$, $x_2 \approx -54.165455$, $x_3 \approx -36.114293$, $x_4 \approx -24.082094$, $x_5 \approx -16.063387$, $x_6 \approx -10.721483$, $x_7 \approx -7.165534$, $x_8 \approx -4.801704$, $x_9 \approx -3.233425$, $x_{10} \approx -2.193674$, $x_{11} \approx -1.496867$, $x_{12} \approx -0.997546$, $x_{13} \approx -0.496305$, $x_{14} \approx -2.894162$, $x_{15} \approx -1.967962$, $x_{16} \approx -1.341355$, $x_{17} \approx -0.870187$, $x_{18} \approx -0.249949$, $x_{19} \approx -1.192219$, $x_{20} \approx -0.731952$, $x_{21} \approx 0.355213$, $x_{22} \approx -1.753322$, $x_{23} \approx -1.189420$, $x_{24} \approx -0.729123$, $x_{25} \approx 0.377844$, $x_{26} \approx -1.937872$, $x_{27} \approx -1.320350$, $x_{28} \approx -0.851919$, $x_{29} \approx -0.200959$, $x_{30} \approx -1.119386$, $x_{31} \approx -0.654291$, $x_{32} \approx 1.547010$, $x_{33} \approx 1.360051$, $x_{34} \approx 1.325828$, $x_{35} \approx 1.324719$, $x_{36} \approx 1.324718 \approx x_{37}$.

(d)

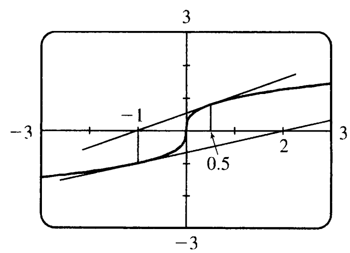


From the figure, we see that the tangent line corresponding to $x_1 = 1$ results in a sequence of approximations that converges quite quickly ($x_5 \approx x_6$). The tangent line corresponding to $x_1 = 0.6$ is close to being horizontal, so x_2 is quite far from the root. But the sequence still converges — just a little more slowly ($x_{12} \approx x_{13}$). Lastly, the tangent line corresponding to $x_1 = 0.57$ is very nearly horizontal, x_2 is farther away from the root, and the sequence takes more iterations to converge ($x_{36} \approx x_{37}$).

33. For $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3}$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

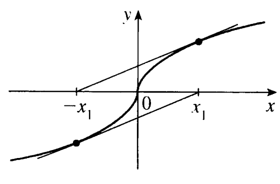
Therefore, each successive approximation becomes twice as large as the previous one in absolute value, so the sequence of approximations fails to converge to the root, which is 0. In the figure, we have $x_1 = 0.5$, $x_2 = -2(0.5) = -1$, and $x_3 = -2(-1) = 2$.



34. According to Newton's Method, for $x_n > 0$,

$$x_{n+1} = x_n - \frac{\sqrt{x_n}}{1/(2\sqrt{x_n})} = x_n - 2x_n = -x_n$$ and for $x_n < 0$,

$$x_{n+1} = x_n - \frac{-\sqrt{-x_n}}{1/(2\sqrt{-x_n})} = x_n - [-2(-x_n)] = -x_n.$$ So we can see that after choosing any value x_1 the subsequent values will alternate between $-x_1$ and x_1 and never approach the root.



35. (a) $f(x) = 3x^4 - 28x^3 + 6x^2 + 24x \quad\Rightarrow\quad f'(x) = 12x^3 - 84x^2 + 12x + 24 \quad\Rightarrow$

$f''(x) = 36x^2 - 168x + 12$. Now to solve $f'(x) = 0$, try $x_1 = \frac{1}{2} \quad\Rightarrow\quad x_2 = x_1 - \dfrac{f'(x_1)}{f''(x_1)} = \dfrac{2}{3} \quad\Rightarrow$

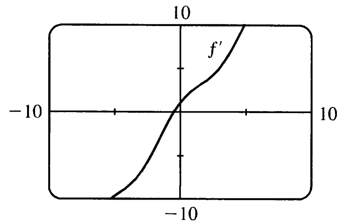
$x_3 \approx 0.6455 \quad\Rightarrow\quad x_4 \approx 0.6452 \quad\Rightarrow\quad x_5 \approx 0.6452$. Now try $x_1 = 6 \quad\Rightarrow\quad x_2 = 7.12 \quad\Rightarrow$

$x_3 \approx 6.8353 \quad\Rightarrow\quad x_4 \approx 6.8102 \quad\Rightarrow\quad x_5 \approx 6.8100$. Finally try $x_1 = -0.5 \quad\Rightarrow\quad x_2 \approx -0.4571 \quad\Rightarrow$

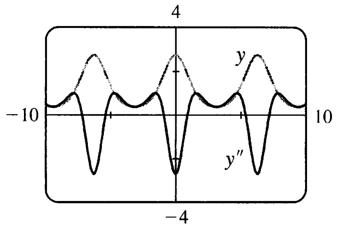
$x_3 \approx -0.4552 \quad\Rightarrow\quad x_4 \approx -0.4552$. Therefore, $x = -0.455$, 6.810 and 0.645 are all critical numbers correct to three decimal places.

(b) $f(-1) = 13$, $f(7) = -1939$, $f(6.810) \approx -1949.07$, $f(-0.455) \approx -6.912$, $f(0.645) \approx 10.982$. Therefore, $f(6.810) \approx -1949.07$ is the absolute minimum correct to two decimal places.

36. $f(x) = x^2 + \sin x \;\Rightarrow\; f'(x) = 2x + \cos x$. $f'(x)$ exists for all x, so to find the minimum of f, we can examine the zeros of f'. From the graph of f', we see that a good choice for x_1 is $x_1 = -0.5$. Use $g(x) = 2x + \cos x$ and $g'(x) = 2 - \sin x$ to obtain $x_2 \approx -0.450627$, $x_3 \approx -0.450184 \approx x_4$. Since $f''(x) = 2 - \sin x > 0$ for all x, $f(-0.450184) \approx -0.232466$ is the absolute minimum.

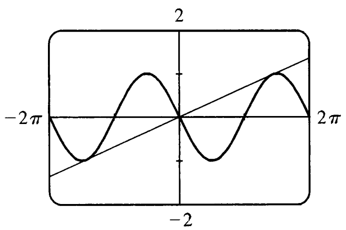


37.



From the figure, we see that $y = f(x) = e^{\cos x}$ is periodic with period 2π. To find the x-coordinates of the IP, we only need to approximate the zeros of y'' on $[0, \pi]$. $f'(x) = -e^{\cos x} \sin x \;\Rightarrow$ $f''(x) = e^{\cos x}(\sin^2 x - \cos x)$. Since $e^{\cos x} \neq 0$, we will use Newton's method with $g(x) = \sin^2 x - \cos x$, $g'(x) = 2\sin x \cos x + \sin x$, and $x_1 = 1$. $x_2 \approx 0.904173$, $x_3 \approx 0.904557 \approx x_4$. Thus, $(0.904557, 1.855277)$ is the IP.

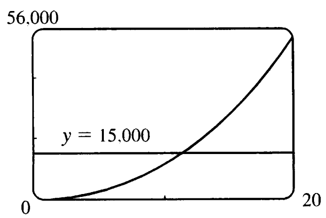
38.



$f(x) = -\sin x \;\Rightarrow\; f'(x) = -\cos x$. At $x = a$, the slope of the tangent line is $f'(a) = -\cos a$. The line through the origin and $(a, f(a))$ is $y = \dfrac{-\sin a - 0}{a - 0}x$. If this line is to be tangent to f at $x = a$, then its slope must equal $f'(a)$. Thus, $\dfrac{-\sin a}{a} = -\cos a \;\Rightarrow\; \tan a = a$.

To solve this equation using Newton's method, let $g(x) = \tan x - x$, $g'(x) = \sec^2 x - 1$, and $x_{n+1} = x_n - \dfrac{\tan x_n - x_n}{\sec^2 x_n - 1}$ with $x_1 = 4.5$ (estimated from the figure). $x_2 \approx 4.493614$, $x_3 \approx 4.493410$, $x_4 \approx 4.493409 \approx x_5$. Thus, the slope of the line that has the largest slope is $f'(x_5) \approx 0.217234$.

39.



The volume of the silo, in terms of its radius, is

$V(r) = \pi r^2(30) + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = 30\pi r^2 + \frac{2}{3}\pi r^3$.

From a graph of V, we see that $V(r) = 15{,}000$ at $r \approx 11$ ft. Now we use Newton's method to solve the equation $V(r) - 15{,}000 = 0$.

$\dfrac{dV}{dr} = 60\pi r + 2\pi r^2$, so $r_{n+1} = r_n - \dfrac{30\pi r_n^2 + \frac{2}{3}\pi r_n^3 - 15{,}000}{60\pi r_n + 2\pi r_n^2}$. Taking $r_1 = 11$, we get $r_2 \approx 11.2853$, $r_3 \approx 11.2807 \approx r_4$. So in order for the silo to hold $15{,}000\ \text{ft}^3$ of grain, its radius must be about 11.2807 ft.

40. Let the radius of the circle be r. Using $s = r\theta$, we have $5 = r\theta$ and so $r = 5/\theta$. From the Law of Cosines we get

$4^2 = r^2 + r^2 - 2 \cdot r \cdot r \cdot \cos\theta \quad \Leftrightarrow \quad 16 = 2r^2(1 - \cos\theta) = 2(5/\theta)^2(1 - \cos\theta)$.

Multiplying by θ^2 gives $16\theta^2 = 50(1 - \cos\theta)$, so we take

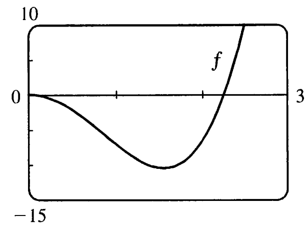
$f(\theta) = 16\theta^2 + 50\cos\theta - 50$ and $f'(\theta) = 32\theta - 50\sin\theta$. The formula

for Newton's method is $\theta_{n+1} = \theta_n - \dfrac{16\theta_n^2 + 50\cos\theta_n - 50}{32\theta_n - 50\sin\theta_n}$. From the

graph of f, we can use $\theta_1 = 2.2$, giving us $\theta_2 \approx 2.2662$,

$\theta_3 \approx 2.2622 \approx \theta_4$. So correct to four decimal places, the angle is

2.2622 radians $\approx 130°$.



41. In this case, $A = 18{,}000$, $R = 375$, and $n = 5(12) = 60$. So the formula $A = \dfrac{R}{i}\left[1 - (1+i)^{-n}\right]$ becomes

$18{,}000 = \dfrac{375}{x}\left[1 - (1+x)^{-60}\right] \quad \Leftrightarrow \quad 48x = 1 - (1+x)^{-60}$ [multiply each term by $(1+x)^{60}$] $\Leftrightarrow$

$48x(1+x)^{60} - (1+x)^{60} + 1 = 0$. Let the LHS be called $f(x)$, so that

$$\begin{aligned} f'(x) &= 48x(60)(1+x)^{59} + 48(1+x)^{60} - 60(1+x)^{59} \\ &= 12(1+x)^{59}\left[4x(60) + 4(1+x) - 5\right] = 12(1+x)^{59}(244x - 1) \end{aligned}$$

$x_{n+1} = x_n - \dfrac{48x_n(1+x_n)^{60} - (1+x_n)^{60} + 1}{12(1+x_n)^{59}(244x_n - 1)}$. An interest rate of 1% per month seems like a reasonable

estimate for $x = i$. So let $x_1 = 1\% = 0.01$, and we get $x_2 \approx 0.0082202$, $x_3 \approx 0.0076802$, $x_4 \approx 0.0076291$, $x_5 \approx 0.0076286 \approx x_6$. Thus, the dealer is charging a monthly interest rate of 0.76286% (or 9.55% per year, compounded monthly).

42. (a) $p(x) = x^5 - (2+r)x^4 + (1+2r)x^3 - (1-r)x^2 + 2(1-r)x + r - 1 \quad \Rightarrow$

$p'(x) = 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1-r)x + 2(1-r)$. So we use

$x_{n+1} = x_n - \dfrac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1-r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1-r)x_n + 2(1-r)}$. We substitute in the value

$r \approx 3.04042 \times 10^{-6}$ in order to evaluate the approximations numerically. The libration point L_1 is slightly less than 1 AU from the Sun, so we take $x_1 = 0.95$ as our first approximation, and get $x_2 \approx 0.96682$, $x_3 \approx 0.97770$, $x_4 \approx 0.98451$, $x_5 \approx 0.98830$, $x_6 \approx 0.98976$, $x_7 \approx 0.98998$, $x_8 \approx 0.98999 \approx x_9$.

So, to five decimal places, L_1 is located 0.98999 AU from the Sun (or 0.01001 AU from Earth).

(b) In this case we use Newton's method with the function

$p(x) - 2rx^2 = x^5 - (2+r)x^4 + (1+2r)x^3 - (1+r)x^2 + 2(1-r)x + r - 1 \quad \Rightarrow$

$\left[p(x) - 2rx^2\right]' = 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1+r)x + 2(1-r)$. So

$x_{n+1} = x_n - \dfrac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1+r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1+r)x_n + 2(1-r)}$. Again, we substitute

$r \approx 3.04042 \times 10^{-6}$. L_2 is slightly more than 1 AU from the Sun and, judging from the result of part (a), probably less than 0.02 AU from Earth. So we take $x_1 = 1.02$ and get $x_2 \approx 1.01422$, $x_3 \approx 1.01118$, $x_4 \approx 1.01018$, $x_5 \approx 1.01008 \approx x_6$. So, to five decimal places, L_2 is located 1.01008 AU from the Sun (or 0.01008 AU from Earth).

4.10 Antiderivatives

1. $f(x) = 6x^2 - 8x + 3 \quad \Rightarrow \quad F(x) = 6\dfrac{x^{2+1}}{2+1} - 8\dfrac{x^{1+1}}{1+1} + 3x + C = 2x^3 - 4x^2 + 3x + C$

Check: $F'(x) = 2 \cdot 3x^2 - 4 \cdot 2x + 3 + 0 = 6x^2 - 8x + 3 = f(x)$

2. $f(x) = 4 + x^2 - 5x^3 \quad \Rightarrow \quad F(x) = 4x + \frac{1}{3}x^3 - \frac{5}{4}x^4 + C$

3. $f(x) = 1 - x^3 + 5x^5 - 3x^7 \quad \Rightarrow \quad F(x) = x - \dfrac{x^{3+1}}{3+1} + 5\dfrac{x^{5+1}}{5+1} - 3\dfrac{x^{7+1}}{7+1} + C = x - \frac{1}{4}x^4 + \frac{5}{6}x^6 - \frac{3}{8}x^8 + C$

4. $f(x) = x^{20} + 4x^{10} + 8 \quad \Rightarrow \quad F(x) = \frac{1}{21}x^{21} + \frac{4}{11}x^{11} + 8x + C$

5. $f(x) = 5x^{1/4} - 7x^{3/4} \quad \Rightarrow \quad F(x) = 5\dfrac{x^{1/4+1}}{\frac{1}{4}+1} - 7\dfrac{x^{3/4+1}}{\frac{3}{4}+1} + C = 5\dfrac{x^{5/4}}{5/4} - 7\dfrac{x^{7/4}}{7/4} + C = 4x^{5/4} - 4x^{7/4} + C$

6. $f(x) = 2x + 3x^{1.7} \quad \Rightarrow \quad F(x) = x^2 + \frac{3}{2.7}x^{2.7} + C = x^2 + \frac{10}{9}x^{2.7} + C$

7. $f(x) = 6\sqrt{x} - \sqrt[6]{x} = 6x^{1/2} - x^{1/6} \quad \Rightarrow$

$$F(x) = 6\,\frac{x^{1/2+1}}{\frac{1}{2}+1} - \frac{x^{1/6+1}}{\frac{1}{6}+1} + C = 6\,\frac{x^{3/2}}{3/2} - \frac{x^{7/6}}{7/6} + C = 4x^{3/2} - \tfrac{6}{7}x^{7/6} + C$$

8. $f(x) = \sqrt[4]{x^3} + \sqrt[3]{x^4} = x^{3/4} + x^{4/3} \quad \Rightarrow \quad F(x) = \dfrac{x^{7/4}}{7/4} + \dfrac{x^{7/3}}{7/3} + C = \frac{4}{7}x^{7/4} + \frac{3}{7}x^{7/3} + C$

9. $f(x) = \dfrac{10}{x^9} = 10x^{-9}$ has domain $(-\infty, 0) \cup (0, \infty)$, so $F(x) = \begin{cases} \dfrac{10x^{-8}}{-8} + C_1 = -\dfrac{5}{4x^8} + C_1 & \text{if } x < 0 \\ -\dfrac{5}{4x^8} + C_2 & \text{if } x > 0 \end{cases}$

See Example 1 for a similar problem.

10. $g(x) = \dfrac{5 - 4x^3 + 2x^6}{x^6} = 5x^{-6} - 4x^{-3} + 2$ has domain $(-\infty, 0) \cup (0, \infty)$, so

$$G(x) = \begin{cases} 5\dfrac{x^{-5}}{-5} - 4\dfrac{x^{-2}}{-2} + 2x + C_1 = -\dfrac{1}{x^5} + \dfrac{2}{x^2} + 2x + C_1 & \text{if } x < 0 \\ -\dfrac{1}{x^5} + \dfrac{2}{x^2} + 2x + C_2 & \text{if } x > 0 \end{cases}$$

11. $f(u) = \dfrac{u^4 + 3\sqrt{u}}{u^2} = \dfrac{u^4}{u^2} + \dfrac{3u^{1/2}}{u^2} = u^2 + 3u^{-3/2} \quad \Rightarrow$

$$F(u) = \frac{u^3}{3} + 3\,\frac{u^{-3/2+1}}{-3/2+1} + C = \frac{1}{3}u^3 + 3\frac{u^{-1/2}}{-1/2} + C = \frac{1}{3}u^3 - \frac{6}{\sqrt{u}} + C$$

12. $f(x) = 3e^x + 7\sec^2 x \quad \Rightarrow \quad F(x) = 3e^x + 7\tan x + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

13. $g(\theta) = \cos\theta - 5\sin\theta \quad \Rightarrow \quad G(\theta) = \sin\theta - 5(-\cos\theta) + C = \sin\theta + 5\cos\theta + C$

14. $h(\theta) = \dfrac{\sin\theta}{\cos^2\theta} = \dfrac{1}{\cos\theta} \cdot \dfrac{\sin\theta}{\cos\theta} = \sec\theta\tan\theta \quad \Rightarrow \quad H(\theta) = \sec\theta + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

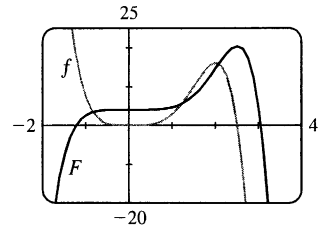
15. $f(x) = 2x + 5(1 - x^2)^{-1/2} = 2x + \dfrac{5}{\sqrt{1 - x^2}} \quad \Rightarrow \quad F(x) = x^2 + 5\sin^{-1} x + C$

16. $f(x) = \dfrac{x^2 + x + 1}{x} = x + 1 + \dfrac{1}{x} \quad \Rightarrow \quad F(x) = \begin{cases} \frac{1}{2}x^2 + x + \ln|x| + C_1 & \text{if } x < 0 \\ \frac{1}{2}x^2 + x + \ln|x| + C_2 & \text{if } x > 0 \end{cases}$

17. $f(x) = 5x^4 - 2x^5 \quad \Rightarrow \quad F(x) = 5 \cdot \dfrac{x^5}{5} - 2 \cdot \dfrac{x^6}{6} + C = x^5 - \frac{1}{3}x^6 + C.$

$F(0) = 4 \quad \Rightarrow \quad 0^5 - \frac{1}{3} \cdot 0^6 + C = 4 \quad \Rightarrow \quad C = 4$, so

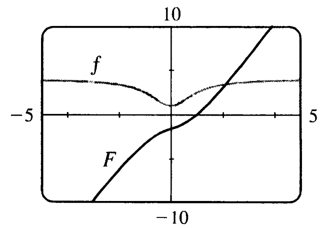
$F(x) = x^5 - \frac{1}{3}x^6 + 4$. The graph confirms our answer since $f(x) = 0$ when F has a local maximum, f is positive when F is increasing, and f is negative when F is decreasing.



18. $f(x) = 4 - 3(1 + x^2)^{-1} = 4 - \dfrac{3}{1 + x^2} \quad \Rightarrow$

$F(x) = 4x - 3\tan^{-1} x + C$. $F(1) = 0 \quad \Rightarrow \quad 4 - 3\left(\frac{\pi}{4}\right) + C = 0 \quad \Rightarrow$

$C = \frac{3\pi}{4} - 4$, so $F(x) = 4x - 3\tan^{-1} x + \frac{3\pi}{4} - 4$. Note that f is positive and F is increasing on $\mathbb{R}$. Also, f has smaller values where the slopes of the tangent lines of F are smaller.



19. $f''(x) = 6x + 12x^2 \quad \Rightarrow \quad f'(x) = 6 \cdot \dfrac{x^2}{2} + 12 \cdot \dfrac{x^3}{3} + C = 3x^2 + 4x^3 + C \quad \Rightarrow$

$f(x) = 3 \cdot \dfrac{x^3}{3} + 4 \cdot \dfrac{x^4}{4} + Cx + D = x^3 + x^4 + Cx + D$ [C and D are just arbitrary constants]

20. $f''(x) = 2 + x^3 + x^6 \quad \Rightarrow \quad f'(x) = 2x + \frac{1}{4}x^4 + \frac{1}{7}x^7 + C \quad \Rightarrow \quad f(x) = x^2 + \frac{1}{20}x^5 + \frac{1}{56}x^8 + Cx + D$

21. $f''(x) = 1 + x^{4/5} \quad \Rightarrow \quad f'(x) = x + \frac{5}{9}x^{9/5} + C \quad \Rightarrow$

$f(x) = \frac{1}{2}x^2 + \frac{5}{9} \cdot \frac{5}{14}x^{14/5} + Cx + D = \frac{1}{2}x^2 + \frac{25}{126}x^{14/5} + Cx + D$

22. $f''(x) = \cos x \quad \Rightarrow \quad f'(x) = \sin x + C \quad \Rightarrow \quad f(x) = -\cos x + Cx + D$

23. $f'''(t) = e^t \quad \Rightarrow \quad f''(t) = e^t + C \quad \Rightarrow \quad f'(t) = e^t + Ct + D \quad \Rightarrow \quad f(t) = e^t + \frac{1}{2}Ct^2 + Dt + E$

24. $f'''(t) = t - \sqrt{t} \quad \Rightarrow \quad f''(t) = \frac{1}{2}t^2 - \frac{2}{3}t^{3/2} + C \quad \Rightarrow \quad f'(t) = \frac{1}{6}t^3 - \frac{4}{15}t^{5/2} + Ct + D \quad \Rightarrow$

$f(t) = \frac{1}{24}t^4 - \frac{8}{105}t^{7/2} + \frac{1}{2}Ct^2 + Dt + E$

25. $f'(x) = 1 - 6x \quad \Rightarrow \quad f(x) = x - 3x^2 + C$. $f(0) = C$ and $f(0) = 8 \quad \Rightarrow \quad C = 8$, so $f(x) = x - 3x^2 + 8$.

26. $f'(x) = 8x^3 + 12x + 3 \quad \Rightarrow \quad f(x) = 2x^4 + 6x^2 + 3x + C$. $f(1) = 11 + C$ and $f(1) = 6 \quad \Rightarrow$

$11 + C = 6 \quad \Rightarrow \quad C = -5$, so $f(x) = 2x^4 + 6x^2 + 3x - 5$.

27. $f'(x) = \sqrt{x}(6 + 5x) = 6x^{1/2} + 5x^{3/2} \quad \Rightarrow \quad f(x) = 4x^{3/2} + 2x^{5/2} + C$.

$f(1) = 6 + C$ and $f(1) = 10 \quad \Rightarrow \quad C = 4$, so $f(x) = 4x^{3/2} + 2x^{5/2} + 4$.

28. $f'(x) = 2x - 3/x^4 = 2x - 3x^{-4} \quad \Rightarrow \quad f(x) = x^2 + x^{-3} + C$ because we're given that $x > 0$.

$f(1) = 2 + C$ and $f(1) = 3 \quad \Rightarrow \quad C = 1$, so $f(x) = x^2 + 1/x^3 + 1$.

29. $f'(t) = 2\cos t + \sec^2 t \;\Rightarrow\; f(t) = 2\sin t + \tan t + C$ because $-\pi/2 < t < \pi/2$.
$f(\frac{\pi}{3}) = 2(\sqrt{3}/2) + \sqrt{3} + C = 2\sqrt{3} + C$ and $f(\frac{\pi}{3}) = 4 \;\Rightarrow\; C = 4 - 2\sqrt{3}$, so
$f(t) = 2\sin t + \tan t + 4 - 2\sqrt{3}$.

30. $f'(x) = 3x^{-2} \;\Rightarrow\; f(x) = \begin{cases} -3/x + C_1 & \text{if } x > 0 \\ -3/x + C_2 & \text{if } x < 0 \end{cases}$ $\quad f(1) = -3 + C_1 = 0 \;\Rightarrow\; C_1 = 3$,
$f(-1) = 3 + C_2 = 0 \;\Rightarrow\; C_2 = -3$. So $f(x) = \begin{cases} -3/x + 3 & \text{if } x > 0 \\ -3/x - 3 & \text{if } x < 0 \end{cases}$

31. $f'(x) = 2/x \;\Rightarrow\; f(x) = 2\ln|x| + C = 2\ln(-x) + C$ (since $x < 0$). Now
$f(-1) = 2\ln 1 + C = 2(0) + C = 7 \;\Rightarrow\; C = 7$. Therefore, $f(x) = 2\ln(-x) + 7$, $x < 0$.

32. $f'(x) = 4/\sqrt{1 - x^2} \;\Rightarrow\; f(x) = 4\sin^{-1} x + C$. $f(\frac{1}{2}) = 4\sin^{-1}(\frac{1}{2}) + C = 4 \cdot \frac{\pi}{6} + C$ and $f(\frac{1}{2}) = 1 \;\Rightarrow$
$\frac{2\pi}{3} + C = 1 \;\Rightarrow\; C = 1 - \frac{2\pi}{3}$, so $f(x) = 4\sin^{-1} x + 1 - \frac{2\pi}{3}$.

33. $f''(x) = 24x^2 + 2x + 10 \;\Rightarrow\; f'(x) = 8x^3 + x^2 + 10x + C$. $f'(1) = 8 + 1 + 10 + C$ and $f'(1) = -3 \;\Rightarrow$
$19 + C = -3 \;\Rightarrow\; C = -22$, so $f'(x) = 8x^3 + x^2 + 10x - 22$ and hence,
$f(x) = 2x^4 + \frac{1}{3}x^3 + 5x^2 - 22x + D$. $f(1) = 2 + \frac{1}{3} + 5 - 22 + D$ and $f(1) = 5 \;\Rightarrow\; D = 22 - \frac{7}{3} = \frac{59}{3}$,
so $f(x) = 2x^4 + \frac{1}{3}x^3 + 5x^2 - 22x + \frac{59}{3}$.

34. $f''(x) = 4 - 6x - 40x^3 \;\Rightarrow\; f'(x) = 4x - 3x^2 - 10x^4 + C$. $f'(0) = C$ and $f'(0) = 1 \;\Rightarrow\; C = 1$, so
$f'(x) = 4x - 3x^2 - 10x^4 + 1$ and hence, $f(x) = 2x^2 - x^3 - 2x^5 + x + D$. $f(0) = D$ and $f(0) = 2 \;\Rightarrow$
$D = 2$, so $f(x) = 2x^2 - x^3 - 2x^5 + x + 2$.

35. $f''(\theta) = \sin\theta + \cos\theta \;\Rightarrow\; f'(\theta) = -\cos\theta + \sin\theta + C$. $f'(0) = -1 + C$ and $f'(0) = 4 \;\Rightarrow\; C = 5$, so
$f'(\theta) = -\cos\theta + \sin\theta + 5$ and hence, $f(\theta) = -\sin\theta - \cos\theta + 5\theta + D$. $f(0) = -1 + D$ and $f(0) = 3 \;\Rightarrow$
$D = 4$, so $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$.

36. $f''(t) = 3/\sqrt{t} = 3t^{-1/2} \;\Rightarrow\; f'(t) = 6t^{1/2} + C$. $f'(4) = 12 + C$ and $f'(4) = 7 \;\Rightarrow\; C = -5$, so
$f'(t) = 6t^{1/2} - 5$ and hence, $f(t) = 4t^{3/2} - 5t + D$. $f(4) = 32 - 20 + D$ and $f(4) = 20 \;\Rightarrow\; D = 8$, so
$f(t) = 4t^{3/2} - 5t + 8$.

37. $f''(x) = 2 - 12x \;\Rightarrow\; f'(x) = 2x - 6x^2 + C \;\Rightarrow\; f(x) = x^2 - 2x^3 + Cx + D$.
$f(0) = D$ and $f(0) = 9 \;\Rightarrow\; D = 9$. $f(2) = 4 - 16 + 2C + 9 = 2C - 3$ and $f(2) = 15 \;\Rightarrow\; 2C = 18 \;\Rightarrow$
$C = 9$, so $f(x) = x^2 - 2x^3 + 9x + 9$.

38. $f''(x) = 20x^3 + 12x^2 + 4 \;\Rightarrow\; f'(x) = 5x^4 + 4x^3 + 4x + C \;\Rightarrow\; f(x) = x^5 + x^4 + 2x^2 + Cx + D$.
$f(0) = D$ and $f(0) = 8 \;\Rightarrow\; D = 8$. $f(1) = 1 + 1 + 2 + C + 8 = C + 12$ and $f(1) = 5 \;\Rightarrow\; C = -7$, so
$f(x) = x^5 + x^4 + 2x^2 - 7x + 8$.

39. $f''(x) = 2 + \cos x \;\Rightarrow\; f'(x) = 2x + \sin x + C \;\Rightarrow\; f(x) = x^2 - \cos x + Cx + D$. $f(0) = -1 + D$ and
$f(0) = -1 \;\Rightarrow\; D = 0$. $f(\frac{\pi}{2}) = \pi^2/4 + (\frac{\pi}{2})C$ and $f(\frac{\pi}{2}) = 0 \;\Rightarrow\; (\frac{\pi}{2})C = -\pi^2/4 \;\Rightarrow\; C = -\frac{\pi}{2}$, so
$f(x) = x^2 - \cos x - (\frac{\pi}{2})x$.

40. $f''(t) = 2e^t + 3\sin t \;\Rightarrow\; f'(t) = 2e^t - 3\cos t + C \;\Rightarrow\; f(t) = 2e^t - 3\sin t + Ct + D$. $f(0) = 2 + D$ and
$f(0) = 0 \;\Rightarrow\; D = -2$. $f(\pi) = 2e^\pi + \pi C - 2$ and $f(\pi) = 0 \;\Rightarrow\; \pi C = 2 - 2e^\pi \;\Rightarrow\; C = \dfrac{2 - 2e^\pi}{\pi}$, so
$f(t) = 2e^t - 3\sin t + \dfrac{2 - 2e^\pi}{\pi}t - 2$.

41. $f''(x) = x^{-2}, x > 0 \;\Rightarrow\; f'(x) = -1/x + C \;\Rightarrow\; f(x) = -\ln|x| + Cx + D = -\ln x + Cx + D$ (since $x > 0$). $f(1) = 0 \;\Rightarrow\; C + D = 0$ and $f(2) = 0 \;\Rightarrow\; -\ln 2 + 2C + D = 0 \;\Rightarrow\; -\ln 2 + 2C - C = 0$ [since $D = -C$] $\Rightarrow\; -\ln 2 + C = 0 \;\Rightarrow\; C = \ln 2$ and $D = -\ln 2$. So $f(x) = -\ln x + (\ln 2)x - \ln 2$.

42. $f'''(x) = \sin x \;\Rightarrow\; f''(x) = -\cos x + C \;\Rightarrow\; 1 = f''(0) = -1 + C \;\Rightarrow\; C = 2$, so $f''(x) = -\cos x + 2 \;\Rightarrow\; f'(x) = -\sin x + 2x + D \;\Rightarrow\; 1 = f'(0) = D \;\Rightarrow\; f'(x) = -\sin x + 2x + 1$ $\Rightarrow\; f(x) = \cos x + x^2 + x + E \;\Rightarrow\; 1 = f(0) = 1 + E \;\Rightarrow\; E = 0$, so $f(x) = \cos x + x^2 + x$.

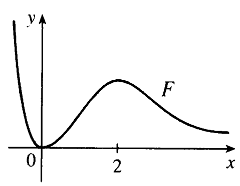
43. Given $f'(x) = 2x + 1$, we have $f(x) = x^2 + x + C$. Since f passes through $(1, 6)$, $f(1) = 6 \;\Rightarrow\; 1^2 + 1 + C = 6 \;\Rightarrow\; C = 4$. Therefore, $f(x) = x^2 + x + 4$ and $f(2) = 2^2 + 2 + 4 = 10$.

44. $f'(x) = x^3 \;\Rightarrow\; f(x) = \frac{1}{4}x^4 + C$. $x + y = 0 \;\Rightarrow\; y = -x \;\Rightarrow\; m = -1$. Now $m = f'(x) \;\Rightarrow$ $-1 = x^3 \;\Rightarrow\; x = -1 \;\Rightarrow\; y = 1$ (from the equation of the tangent line), so $(-1, 1)$ is a point on the graph of f. From f, $1 = \frac{1}{4}(-1)^4 + C \;\Rightarrow\; C = \frac{3}{4}$. Therefore, the function is $f(x) = \frac{1}{4}x^4 + \frac{3}{4}$.

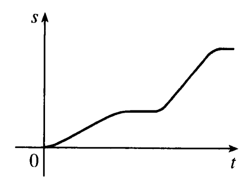
45. b is the antiderivative of f. For small x, f is negative, so the graph of its antiderivative must be decreasing. But both a and c are increasing for small x, so only b can be f's antiderivative. Also, f is positive where b is increasing, which supports our conclusion.

46. We know right away that c cannot be f's antiderivative, since the slope of c is not zero at the x-value where $f = 0$. Now f is positive when a is increasing and negative when a is decreasing, so a is the antiderivative of f.

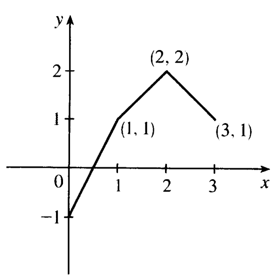
47. The graph of F will have a minimum at 0 and a maximum at 2, since $f = F'$ goes from negative to positive at $x = 0$, and from positive to negative at $x = 2$.



48. The position function is the antiderivative of the velocity function, so its graph has to be horizontal where the velocity function is equal to 0.



49.



$$f'(x) = \begin{cases} 2 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 < x < 2 \\ -1 & \text{if } 2 < x \le 3 \end{cases} \;\Rightarrow\; f(x) = \begin{cases} 2x + C & \text{if } 0 \le x < 1 \\ x + D & \text{if } 1 < x < 2 \\ -x + E & \text{if } 2 < x \le 3 \end{cases}$$

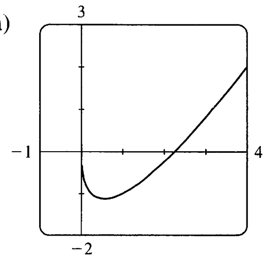
$f(0) = -1 \;\Rightarrow\; 2(0) + C = -1 \;\Rightarrow\; C = -1$. Starting at the point $(0, -1)$ and moving to the right on a line with slope 2 gets us to the point $(1, 1)$. The slope for $1 < x < 2$ is 1, so we get to the point $(2, 2)$. Here we have used the fact that f is continuous. We can include the point $x = 1$ on either the first or the second part of f. The line connecting $(1, 1)$ to $(2, 2)$ is $y = x$, so $D = 0$. The slope for

$2 < x \leq 3$ is -1, so we get to $(3, 1)$. $f(3) = 1 \Rightarrow -3 + E = 1 \Rightarrow E = 4$. Thus,

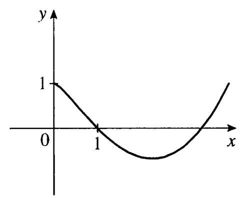
$$f(x) = \begin{cases} 2x - 1 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x < 2 \\ -x + 4 & \text{if } 2 \leq x \leq 3 \end{cases}$$

Note that $f'(x)$ does not exist at $x = 1$ or at $x = 2$.

50. (a)

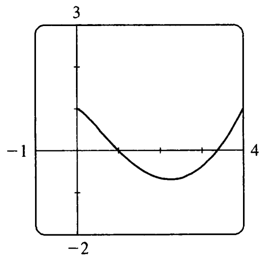


(b) Since $F(0) = 1$, we can start our graph at $(0, 1)$. f has a minimum at about $x = 0.5$, so its derivative is zero there. f is decreasing on $(0, 0.5)$, so its derivative is negative and hence, F is CD on $(0, 0.5)$ and has an IP at $x \approx 0.5$. On $(0.5, 2.2)$, f is negative and increasing (f' is positive), so F is decreasing and CU. On $(2.2, \infty)$, f is positive and increasing, so F is increasing and CU.

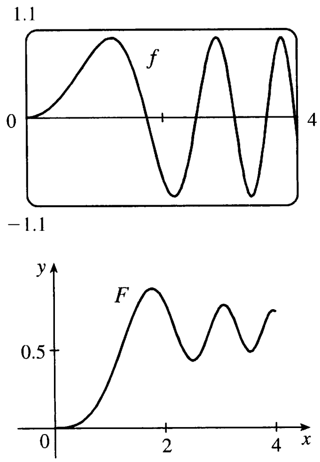


(c) $f(x) = 2x - 3\sqrt{x} \Rightarrow$
$F(x) = x^2 - 3 \cdot \frac{2}{3}x^{3/2} + C$. $F(0) = C$ and
$F(0) = 1 \Rightarrow C = 1$, so
$F(x) = x^2 - 2x^{3/2} + 1$.

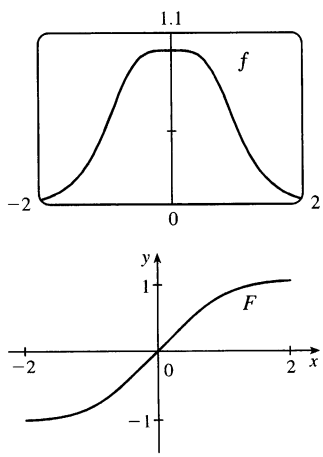
(d)



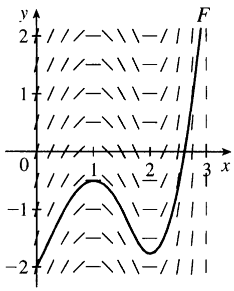
51. $f(x) = \sin(x^2)$, $0 \leq x \leq 4$



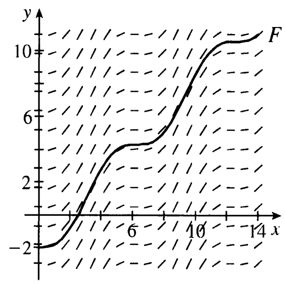
52. $f(x) = 1/(x^4 + 1)$



53.



54.

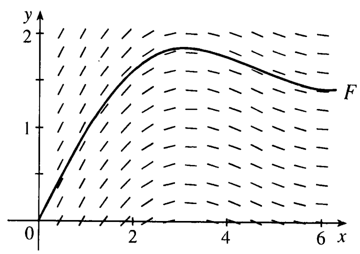


55.

x	$f(x)$
0	1
0.5	0.959
1.0	0.841
1.5	0.665
2.0	0.455
2.5	0.239
3.0	0.047

x	$f(x)$
3.5	−0.100
4.0	−0.189
4.5	−0.217
5.0	−0.192
5.5	−0.128
6.0	−0.047

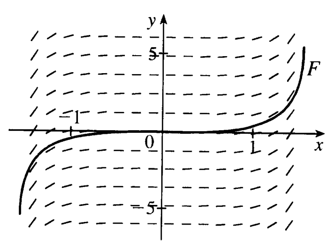
We compute slopes [values of $f(x) = (\sin x)/x$ for $0 < x < 2\pi$] as in the table [$\lim_{x\to 0^+} f(x) = 1$] and draw a direction field as in Example 6. Then we use the direction field to graph F starting at $(0, 0)$.



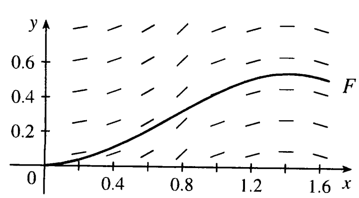
56.

x	$f(x)$
0	0
±0.2	0.041
±0.4	0.169
±0.6	0.410
±0.8	0.824
±1.0	1.557
±1.2	3.087
±1.4	8.117
±1.5	21.152

We compute slopes [values of $f(x) = x\tan x$ for $-\pi/2 < x < \pi/2$] as in the table and draw a direction field as in Example 6. Then we use the direction field to graph F starting at $(0, 0)$ and extending in both directions. Note that if f is an even function, then the antiderivative F that passes through the origin is an odd function.

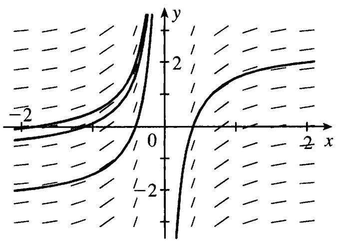


57.



Remember that the given table values of f are the slopes of F at any x. For example, at $x = 1.4$, the slope of F is $f(1.4) = 0$.

58. (a)



(b) The general antiderivative of $f(x) = x^{-2}$ is

$$F(x) = \begin{cases} -1/x + C_1 & \text{if } x < 0 \\ -1/x + C_2 & \text{if } x > 0 \end{cases}$$ since $f(x)$ is not defined

at $x = 0$. The graph of the general antiderivatives of $f(x)$ looks like the graph in part (a), as expected.

59. $v(t) = s'(t) = \sin t - \cos t \quad\Rightarrow\quad s(t) = -\cos t - \sin t + C$. $s(0) = -1 + C$ and $s(0) = 0 \quad\Rightarrow\quad C = 1$, so $s(t) = -\cos t - \sin t + 1$.

60. $v(t) = s'(t) = 1.5\sqrt{t} \quad\Rightarrow\quad s(t) = t^{3/2} + C$. $s(4) = 8 + C$ and $s(4) = 10 \quad\Rightarrow\quad C = 2$, so $s(t) = t^{3/2} + 2$.

61. $a(t) = v'(t) = t - 2 \quad\Rightarrow\quad v(t) = \frac{1}{2}t^2 - 2t + C$. $v(0) = C$ and $v(0) = 3 \quad\Rightarrow\quad C = 3$, so $v(t) = \frac{1}{2}t^2 - 2t + 3$ and $s(t) = \frac{1}{6}t^3 - t^2 + 3t + D$. $s(0) = D$ and $s(0) = 1 \quad\Rightarrow\quad D = 1$, and $s(t) = \frac{1}{6}t^3 - t^2 + 3t + 1$.

62. $a(t) = v'(t) = \cos t + \sin t \quad\Rightarrow\quad v(t) = \sin t - \cos t + C \quad\Rightarrow\quad 5 = v(0) = -1 + C \quad\Rightarrow\quad C = 6$, so $v(t) = \sin t - \cos t + 6 \quad\Rightarrow\quad s(t) = -\cos t - \sin t + 6t + D \quad\Rightarrow\quad 0 = s(0) = -1 + D \quad\Rightarrow\quad D = 1$, so $s(t) = -\cos t - \sin t + 6t + 1$.

63. $a(t) = v'(t) = 10\sin t + 3\cos t \quad\Rightarrow\quad v(t) = -10\cos t + 3\sin t + C \quad\Rightarrow$
$s(t) = -10\sin t - 3\cos t + Ct + D$. $s(0) = -3 + D = 0$ and $s(2\pi) = -3 + 2\pi C + D = 12 \quad\Rightarrow\quad D = 3$ and $C = \frac{6}{\pi}$. Thus, $s(t) = -10\sin t - 3\cos t + \frac{6}{\pi}t + 3$.

64. $a(t) = v'(t) = 10 + 3t - 3t^2 \quad\Rightarrow\quad v(t) = 10t + \frac{3}{2}t^2 - t^3 + C \quad\Rightarrow\quad s(t) = 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 + Ct + D \quad\Rightarrow$
$0 = s(0) = D$ and $10 = s(2) = 20 + 4 - 4 + 2C \quad\Rightarrow\quad C = -5$, so $s(t) = -5t + 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4$.

65. (a) We first observe that since the stone is dropped 450 m above the ground, $v(0) = 0$ and $s(0) = 450$.
$v'(t) = a(t) = -9.8 \quad\Rightarrow\quad v(t) = -9.8t + C$. Now $v(0) = 0 \quad\Rightarrow\quad C = 0$, so $v(t) = -9.8t \quad\Rightarrow$
$s(t) = -4.9t^2 + D$. Last, $s(0) = 450 \quad\Rightarrow\quad D = 450 \quad\Rightarrow\quad s(t) = 450 - 4.9t^2$.

(b) The stone reaches the ground when $s(t) = 0$. $450 - 4.9t^2 = 0 \quad\Rightarrow\quad t^2 = 450/4.9 \quad\Rightarrow$
$t_1 = \sqrt{450/4.9} \approx 9.58$ s.

(c) The velocity with which the stone strikes the ground is $v(t_1) = -9.8\sqrt{450/4.9} \approx -93.9$ m/s.

(d) This is just reworking parts (a) and (b) with $v(0) = -5$. Using $v(t) = -9.8t + C$, $v(0) = -5 \quad\Rightarrow$
$0 + C = -5 \quad\Rightarrow\quad v(t) = -9.8t - 5$. So $s(t) = -4.9t^2 - 5t + D$ and $s(0) = 450 \quad\Rightarrow\quad D = 450 \quad\Rightarrow$
$s(t) = -4.9t^2 - 5t + 450$. Solving $s(t) = 0$ by using the quadratic formula gives us
$t = \left(5 \pm \sqrt{8845}\right)/(-9.8) \quad\Rightarrow\quad t_1 \approx 9.09$ s.

66. $v'(t) = a(t) = a \quad\Rightarrow\quad v(t) = at + C$ and $v_0 = v(0) = C \quad\Rightarrow\quad v(t) = at + v_0 \quad\Rightarrow$
$s(t) = \frac{1}{2}at^2 + v_0 t + D \quad\Rightarrow\quad s_0 = s(0) = D \quad\Rightarrow\quad s(t) = \frac{1}{2}at^2 + v_0 t + s_0$

67. By Exercise 66 with $a = -9.8$, $s(t) = -4.9t^2 + v_0 t + s_0$ and $v(t) = s'(t) = -9.8t + v_0$. So
$[v(t)]^2 = (-9.8t + v_0)^2 = (9.8)^2 t^2 - 19.6 v_0 t + v_0^2 = v_0^2 + 96.04t^2 - 19.6 v_0 t = v_0^2 - 19.6\left(-4.9t^2 + v_0 t\right)$.
But $-4.9t^2 + v_0 t$ is just $s(t)$ without the s_0 term; that is, $s(t) - s_0$. Thus, $[v(t)]^2 = v_0^2 - 19.6\,[s(t) - s_0]$.

68. For the first ball, $s_1(t) = -16t^2 + 48t + 432$ from Example 8. For the second ball, $a(t) = -32 \Rightarrow$ $v(t) = -32t + C$, but $v(1) = -32(1) + C = 24 \Rightarrow C = 56$, so $v(t) = -32t + 56 \Rightarrow$ $s(t) = -16t^2 + 56t + D$, but $s(1) = -16(1)^2 + 56(1) + D = 432 \Rightarrow D = 392$, and $s_2(t) = -16t^2 + 56t + 392$. The balls pass each other when $s_1(t) = s_2(t) \Rightarrow$ $-16t^2 + 48t + 432 = -16t^2 + 56t + 392 \Leftrightarrow 8t = 40 \Leftrightarrow t = 5$ s.

Another solution: From Exercise 66, we have $s_1(t) = -16t^2 + 48t + 432$ and $s_2(t) = -16t^2 + 24t + 432$. We now want to solve $s_1(t) = s_2(t-1) \Rightarrow -16t^2 + 48t + 432 = -16(t-1)^2 + 24(t-1) + 432 \Rightarrow$ $48t = 32t - 16 + 24t - 24 \Rightarrow 40 = 8t \Rightarrow t = 5$ s.

69. Using Exercise 66 with $a = -32$, $v_0 = 0$, and $s_0 = h$ (the height of the cliff), we know that the height at time t is $s(t) = -16t^2 + h$. $v(t) = s'(t) = -32t$ and $v(t) = -120 \Rightarrow -32t = -120 \Rightarrow t = 3.75$, so $0 = s(3.75) = -16(3.75)^2 + h \Rightarrow h = 16(3.75)^2 = 225$ ft.

70. (a) $EIy'' = mg(L-x) + \frac{1}{2}\rho g(L-x)^2 \Rightarrow EIy' = -\frac{1}{2}mg(L-x)^2 - \frac{1}{6}\rho g(L-x)^3 + C \Rightarrow$ $EIy = \frac{1}{6}mg(L-x)^3 + \frac{1}{24}\rho g(L-x)^4 + Cx + D$. Since the left end of the board is fixed, we must have $y = y' = 0$ when $x = 0$. Thus, $0 = -\frac{1}{2}mgL^2 - \frac{1}{6}\rho gL^3 + C$ and $0 = \frac{1}{6}mgL^3 + \frac{1}{24}\rho gL^4 + D$. It follows that $EIy = \frac{1}{6}mg(L-x)^3 + \frac{1}{24}\rho g(L-x)^4 + \left(\frac{1}{2}mgL^2 + \frac{1}{6}\rho gL^3\right)x - \left(\frac{1}{6}mgL^3 + \frac{1}{24}\rho gL^4\right)$ and

$$f(x) = y = \frac{1}{EI}\left[\tfrac{1}{6}mg(L-x)^3 + \tfrac{1}{24}\rho g(L-x)^4 + \left(\tfrac{1}{2}mgL^2 + \tfrac{1}{6}\rho gL^3\right)x - \left(\tfrac{1}{6}mgL^3 + \tfrac{1}{24}\rho gL^4\right)\right]$$

(b) $f(L) < 0$, so the end of the board is a *distance* approximately $-f(L)$ below the horizontal. From our result in (a), we calculate

$$\begin{aligned} -f(L) &= \frac{-1}{EI}\left[\tfrac{1}{2}mgL^3 + \tfrac{1}{6}\rho gL^4 - \tfrac{1}{6}mgL^3 - \tfrac{1}{24}\rho gL^4\right] \\ &= \frac{-1}{EI}\left(\tfrac{1}{3}mgL^3 + \tfrac{1}{8}\rho gL^4\right) = -\frac{gL^3}{EI}\left(\frac{m}{3} + \frac{\rho L}{8}\right) \end{aligned}$$

Note: This is positive because g is negative.

71. Marginal cost $= 1.92 - 0.002x = C'(x) \Rightarrow C(x) = 1.92x - 0.001x^2 + K$. But $C(1) = 1.92 - 0.001 + K = 562 \Rightarrow K = 560.081$. Therefore, $C(x) = 1.92x - 0.001x^2 + 560.081 \Rightarrow$ $C(100) = 742.081$, so the cost of producing 100 items is \$742.08.

72. Let the mass, measured from one end, be $m(x)$. Then $m(0) = 0$ and $\rho = \dfrac{dm}{dx} = x^{-1/2} \Rightarrow m(x) = 2x^{1/2} + C$ and $m(0) = C = 0$, so $m(x) = 2\sqrt{x}$. Thus, the mass of the 100-centimeter rod is $m(100) = 2\sqrt{100} = 20$ g.

73. Taking the upward direction to be positive we have that for $0 \le t \le 10$ (using the subscript 1 to refer to $0 \le t \le 10$), $a_1(t) = -(9 - 0.9t) = v_1'(t) \Rightarrow v_1(t) = -9t + 0.45t^2 + v_0$, but $v_1(0) = v_0 = -10 \Rightarrow$ $v_1(t) = -9t + 0.45t^2 - 10 = s_1'(t) \Rightarrow s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + s_0$. But $s_1(0) = 500 = s_0 \Rightarrow$ $s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + 500$. $s_1(10) = -450 + 150 - 100 + 500 = 100$, so it takes more than 10 seconds for the raindrop to fall. Now for $t > 10$, $a(t) = 0 = v'(t) \Rightarrow$ $v(t) = \text{constant} = v_1(10) = -9(10) + 0.45(10)^2 - 10 = -55 \Rightarrow v(t) = -55$. At 55 ft/s, it will take $100/55 \approx 1.8$ s to fall the last 100 ft. Hence, the total time is $10 + \frac{100}{55} = \frac{130}{11} \approx 11.8$ s.

74. $v'(t) = a(t) = -22$. The initial velocity is 50 mi/h $= \frac{50 \cdot 5280}{3600} = \frac{220}{3}$ ft/s, so $v(t) = -22t + \frac{220}{3}$. The car stops when $v(t) = 0 \Leftrightarrow t = \frac{220}{3 \cdot 22} = \frac{10}{3}$. Since $s(t) = -11t^2 + \frac{220}{3}t$, the distance covered is $s\left(\frac{10}{3}\right) = -11\left(\frac{10}{3}\right)^2 + \frac{220}{3} \cdot \frac{10}{3} = \frac{1100}{9} = 122.\overline{2}$ ft.

75. $a(t) = k$, the initial velocity is 30 mi/h $= 30 \cdot \frac{5280}{3600} = 44$ ft/s, and the final velocity (after 5 seconds) is 50 mi/h $= 50 \cdot \frac{5280}{3600} = \frac{220}{3}$ ft/s. So $v(t) = kt + C$ and $v(0) = 44 \Rightarrow C = 44$. Thus, $v(t) = kt + 44 \Rightarrow v(5) = 5k + 44$. But $v(5) = \frac{220}{3}$, so $5k + 44 = \frac{220}{3} \Rightarrow 5k = \frac{88}{3} \Rightarrow k = \frac{88}{15} \approx 5.87\ \text{ft/s}^2$.

76. $a(t) = -16 \Rightarrow v(t) = -16t + v_0$ where v_0 is the car's speed (in ft/s) when the brakes were applied. The car stops when $-16t + v_0 = 0 \Leftrightarrow t = \frac{1}{16}v_0$. Now $s(t) = \frac{1}{2}(-16)t^2 + v_0 t = -8t^2 + v_0 t$. The car travels 200 ft in the time that it takes to stop, so $s\left(\frac{1}{16}v_0\right) = 200 \Rightarrow 200 = -8\left(\frac{1}{16}v_0\right)^2 + v_0\left(\frac{1}{16}v_0\right) = \frac{1}{32}v_0^2 \Rightarrow v_0^2 = 32 \cdot 200 = 6400 \Rightarrow v_0 = 80$ ft/s ($54.\overline{54}$ mi/h).

77. Let the acceleration be $a(t) = k\ \text{km/h}^2$. We have $v(0) = 100$ km/h and we can take the initial position $s(0)$ to be 0. We want the time t_f for which $v(t) = 0$ to satisfy $s(t) < 0.08$ km. In general, $v'(t) = a(t) = k$, so $v(t) = kt + C$, where $C = v(0) = 100$. Now $s'(t) = v(t) = kt + 100$, so $s(t) = \frac{1}{2}kt^2 + 100t + D$, where $D = s(0) = 0$. Thus, $s(t) = \frac{1}{2}kt^2 + 100t$. Since $v(t_f) = 0$, we have $kt_f + 100 = 0$ or $t_f = -100/k$, so

$$s(t_f) = \frac{1}{2}k\left(-\frac{100}{k}\right)^2 + 100\left(-\frac{100}{k}\right) = 10{,}000\left(\frac{1}{2k} - \frac{1}{k}\right) = -\frac{5{,}000}{k}.$$ The condition $s(t_f)$ must satisfy is

$$-\frac{5{,}000}{k} < 0.08 \quad\Rightarrow\quad -\frac{5{,}000}{0.08} > k \quad [k \text{ is negative}] \quad\Rightarrow\quad k < -62{,}500\ \text{km/h}^2,$$ or equivalently, $k < -\frac{3125}{648} \approx -4.82\ \text{m/s}^2$.

78. (a) For $0 \le t \le 3$ we have $a(t) = 60t \Rightarrow v(t) = 30t^2 + C \Rightarrow v(0) = 0 = C \Rightarrow v(t) = 30t^2$, so $s(t) = 10t^3 + C \Rightarrow s(0) = 0 = C \Rightarrow s(t) = 10t^3$. Note that $v(3) = 270$ and $s(3) = 270$.

For $3 < t \le 17$: $a(t) = -g = -32$ ft/s $\Rightarrow v(t) = -32(t-3) + C \Rightarrow v(3) = 270 = C \Rightarrow v(t) = -32(t-3) + 270 \Rightarrow s(t) = -16(t-3)^2 + 270(t-3) + C \Rightarrow s(3) = 270 = C \Rightarrow s(t) = -16(t-3)^2 + 270(t-3) + 270$. Note that $v(17) = -178$ and $s(17) = 914$.

For $17 < t \le 22$: The velocity increases linearly from -178 ft/s to -18 ft/s during this period, so $\dfrac{\Delta v}{\Delta t} = \dfrac{-18 - (-178)}{22 - 17} = \dfrac{160}{5} = 32$. Thus, $v(t) = 32(t-17) - 178 \Rightarrow s(t) = 16(t-17)^2 - 178(t-17) + 914$ and $s(22) = 424$ ft.

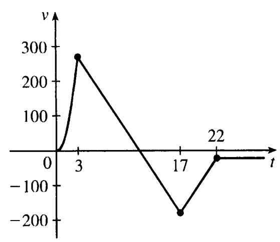
For $t > 22$: $v(t) = -18 \Rightarrow s(t) = -18(t-22) + C$. But $s(22) = 424 = C \Rightarrow s(t) = -18(t-22) + 424$.

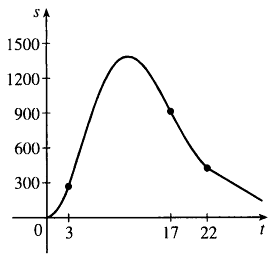
Therefore, until the rocket lands, we have

$$v(t) = \begin{cases} 30t^2 & \text{if } 0 \le t \le 3 \\ -32\,(t-3) + 270 & \text{if } 3 < t \le 17 \\ 32(t-17) - 178 & \text{if } 17 < t \le 22 \\ -18 & \text{if } t > 22 \end{cases}$$

and

$$s(t) = \begin{cases} 10t^3 & \text{if } 0 \le t \le 3 \\ -16(t-3)^2 + 270(t-3) + 270 & \text{if } 3 < t \le 17 \\ 16(t-17)^2 - 178\,(t-17) + 914 & \text{if } 17 < t \le 22 \\ -18(t-22) + 424 & \text{if } t > 22 \end{cases}$$





(b) To find the maximum height, set $v(t)$ on $3 < t \leq 17$ equal to 0. $-32(t-3)+270=0 \Rightarrow t_1 = 11.4375$ s and the maximum height is $s(t_1) = -16(t_1-3)^2 + 270(t_1-3)+270 = 1409.0625$ ft.

(c) To find the time to land, set $s(t) = -18(t-22)+424 = 0$. Then $t - 22 = \frac{424}{18} = 23.\overline{5}$, so $t \approx 45.6$ s.

79. (a) First note that 90 mi/h $= 90 \times \frac{5280}{3600}$ ft/s $= 132$ ft/s. Then $a(t) = 4$ ft/s^2 $\Rightarrow$ $v(t) = 4t + C$, but $v(0) = 0$ $\Rightarrow$ $C = 0$. Now $4t = 132$ when $t = \frac{132}{4} = 33$ s, so it takes 33 s to reach 132 ft/s. Therefore, taking $s(0) = 0$, we have $s(t) = 2t^2$, $0 \leq t \leq 33$. So $s(33) = 2178$ ft. 15 minutes $= 15(60) = 900$ s, so for $33 < t \leq 933$ we have $v(t) = 132$ ft/s $\Rightarrow$ $s(933) = 132(900) + 2178 = 120{,}978$ ft $= 22.9125$ mi.

(b) As in part (a), the train accelerates for 33 s and travels 2178 ft while doing so. Similarly, it decelerates for 33 s and travels 2178 ft at the end of its trip. During the remaining $900 - 66 = 834$ s it travels at 132 ft/s, so the distance traveled is $132 \cdot 834 = 110{,}088$ ft. Thus, the total distance is
$2178 + 110{,}088 + 2178 = 114{,}444$ ft $= 21.675$ mi.

(c) 45 mi $= 45(5280) = 237{,}600$ ft. Subtract $2(2178)$ to take care of the speeding up and slowing down, and we have 233,244 ft at 132 ft/s for a trip of $233{,}244/132 = 1767$ s at 90 mi/h. The total time is
$1767 + 2(33) = 1833$ s $= 30$ min 33 s $= 30.55$ min.

(d) $37.5(60) = 2250$ s. $2250 - 2(33) = 2184$ s at maximum speed. $2184(132) + 2(2178) = 292{,}644$ total feet or $292{,}644/5280 = 55.425$ mi.

4 Review

CONCEPT CHECK

1. A function f has an **absolute maximum** at $x = c$ if $f(c)$ is the largest function value on the entire domain of f, whereas f has a **local maximum** at c if $f(c)$ is the largest function value when x is near c. See Figure 4 in Section 4.1.

2. (a) See Theorem 4.1.3.

(b) See the Closed Interval Method before Example 8 in Section 4.1.

3. (a) See Theorem 4.1.4.

(b) See Definition 4.1.6.

4. (a) See Rolle's Theorem at the beginning of Section 4.2.

(b) See the Mean Value Theorem in Section 4.2. Geometric interpretation—there is some point P on the graph of a function f [on the interval (a, b)] where the tangent line is parallel to the secant line that connects $(a, f(a))$ and $(b, f(b))$.

5. (a) See the I/D Test before Example 1 in Section 4.3.

(b) See the Concavity Test before Example 4 in Section 4.3.

6. (a) See the First Derivative Test after Example 1 in Section 4.3.

(b) See the Second Derivative Test before Example 6 in Section 4.3.

(c) See the note before Example 7 in Section 4.3.

7. (a) See l'Hospital's Rule and the three notes that follow it in Section 4.4.

(b) Write fg as $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$.

(c) Convert the difference into a quotient using a common denominator, rationalizing, factoring, or some other method.

(d) Convert the power to a product by taking the natural logarithm of both sides of $y = f^g$ or by writing f^g as $e^{g \ln f}$.

8. Without calculus you could get misleading graphs that fail to show the most interesting features of a function. See the discussion following Figure 3 in Section 4.5 and the first paragraph in Section 4.6.

9. (a) See Figure 3 in Section 4.9.

(b) $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

(c) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

(d) Newton's method is likely to fail or to work very slowly when $f'(x_1)$ is close to 0.

10. (a) See the definition at the beginning of Section 4.10.

(b) If F_1 and F_2 are both antiderivatives of f on an interval I, then they differ by a constant.

TRUE-FALSE QUIZ

1. False. For example, take $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$, but $f(0) = 0$ is not a maximum or minimum; $(0, 0)$ is an inflection point.

2. False. For example, $f(x) = |x|$ has an absolute minimum at 0, but $f'(0)$ does not exist.

3. False. For example, $f(x) = x$ is continuous on $(0, 1)$ but attains neither a maximum nor a minimum value on $(0, 1)$. Don't confuse this with f being continuous on the *closed* interval $[a, b]$, which would make the statement true.

4. True. By the Mean Value Theorem, $f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{0}{2} = 0$. Note that $|c| < 1 \quad \Leftrightarrow \quad c \in (-1, 1)$.

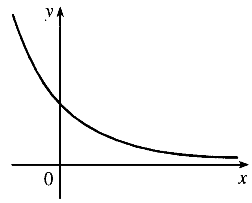
5. True. This is an example of part (b) of the I/D Test.

6. False. For example, the curve $y = f(x) = 1$ has no inflection points but $f''(c) = 0$ for all c.

7. False. $f'(x) = g'(x) \quad \Rightarrow \quad f(x) = g(x) + C$. For example, if $f(x) = x + 2$ and $g(x) = x + 1$, then $f'(x) = g'(x) = 1$, but $f(x) \neq g(x)$.

8. False. Assume there is a function f such that $f(1) = -2$ and $f(3) = 0$. Then by the Mean Value Theorem there exists a number $c \in (1, 3)$ such that $f'(c) = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{0 - (-2)}{2} = 1$. But $f'(x) > 1$ for all x, a contradiction.

9. True. The graph of one such function is sketched.



10. False. At any point $(a, f(a))$, we know that $f'(a) < 0$. So since the tangent line at $(a, f(a))$ is not horizontal, it must cross the x-axis—at $x = b$, say. But since $f''(x) > 0$ for all x, the graph of f must lie above all of its tangents; in particular, $f(b) > 0$. But this is a contradiction, since we are given that $f(x) < 0$ for all x.

11. True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$ (since f and g are increasing on I), so $(f+g)(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = (f+g)(x_2)$.

12. False. $f(x) = x$ and $g(x) = 2x$ are both increasing on $(0, 1)$, but $f(x) - g(x) = -x$ is not increasing on $(0, 1)$.

13. False. Take $f(x) = x$ and $g(x) = x - 1$. Then both f and g are increasing on $(0, 1)$. But $f(x)g(x) = x(x-1)$ is not increasing on $(0, 1)$.

14. True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $0 < f(x_1) < f(x_2)$ and $0 < g(x_1) < g(x_2)$ (since f and g are both positive and increasing). Hence, $f(x_1)\,g(x_1) < f(x_2)\,g(x_1) < f(x_2)\,g(x_2)$. So fg is increasing on I.

15. True. Let $x_1, x_2 \in I$ and $x_1 < x_2$. Then $f(x_1) < f(x_2)$ (f is increasing) $\Rightarrow$
$\dfrac{1}{f(x_1)} > \dfrac{1}{f(x_2)}$ (f is positive) $\Rightarrow$ $g(x_1) > g(x_2)$ $\Rightarrow$ $g(x) = 1/f(x)$ is decreasing on I.

16. False. The most general antiderivative is $F(x) = -1/x + C_1$ for $x < 0$ and $F(x) = -1/x + C_2$ for $x > 0$ (see Example 1 in Section 4.10).

17. True. By the Mean Value Theorem, there exists a number c in $(0, 1)$ such that
$f(1) - f(0) = f'(c)(1 - 0) = f'(c)$. Since $f'(c)$ is nonzero, $f(1) - f(0) \neq 0$, so $f(1) \neq f(0)$.

18. False. $\displaystyle\lim_{x\to 0} \frac{x}{e^x} = \frac{\lim\limits_{x\to 0} x}{\lim\limits_{x\to 0} e^x} = \frac{0}{1} = 0$, not 1.

EXERCISES

1. $f(x) = 10 + 27x - x^3$, $0 \le x \le 4$. $f'(x) = 27 - 3x^2 = -3(x^2 - 9) = -3(x+3)(x-3) = 0$ only when $x = 3$ (since -3 is not in the domain). $f'(x) > 0$ for $x < 3$ and $f'(x) < 0$ for $x > 3$, so $f(3) = 64$ is a local maximum value. Checking the endpoints, we find $f(0) = 10$ and $f(4) = 54$. Thus, $f(0) = 10$ is the absolute minimum value and $f(3) = 64$ is the absolute maximum value.

2. $f(x) = x - \sqrt{x}$, $0 \le x \le 4$. $f'(x) = 1 - 1/(2\sqrt{x}) = 0$ $\Leftrightarrow$ $2\sqrt{x} = 1$ $\Rightarrow$ $x = \frac{1}{4}$. $f'(x)$ does not exist $\Leftrightarrow$ $x = 0$. $f'(x) < 0$ for $0 < x < \frac{1}{4}$ and $f'(x) > 0$ for $\frac{1}{4} < x < 4$, so $f\left(\frac{1}{4}\right) = -\frac{1}{4}$ is a local and absolute minimum value. $f(0) = 0$ and $f(4) = 2$, so $f(4) = 2$ is the absolute maximum value.

3. $f(x) = \dfrac{x}{x^2 + x + 1}$, $-2 \le x \le 0$. $f'(x) = \dfrac{(x^2 + x + 1)(1) - x(2x + 1)}{(x^2 + x + 1)^2} = \dfrac{1 - x^2}{(x^2 + x + 1)^2} = 0$ $\Leftrightarrow$ $x = -1$ (since 1 is not in the domain). $f'(x) < 0$ for $-2 < x < -1$ and $f'(x) > 0$ for $-1 < x < 0$, so $f(-1) = -1$ is a local and absolute minimum value. $f(-2) = -\frac{2}{3}$ and $f(0) = 0$, so $f(0) = 0$ is an absolute maximum value.

4. $f(x) = (x^2 + 2x)^3$, $[-2, 1]$. $f'(x) = 3(x^2 + 2x)^2(2x + 2) = 6(x + 1)x^2(x + 2)^2$, so the only critical numbers in the interior of the domain are $x = -1, 0$. $f'(x) < 0$ for $-2 < x < -1$ and $f'(x) > 0$ for $-1 < x < 0$ and $0 < x < 1$, so f is decreasing on $(-2, -1)$ and increasing on $(-1, 1)$. Thus, $f(-1) = -1$ is a local minimum value. $f(-2) = 0$ and $f(1) = 27$, so the local minimum value is the absolute minimum value and $f(1) = 27$ is the absolute maximum value.

5. $f(x) = x + \sin 2x$, $[0, \pi]$. $f'(x) = 1 + 2\cos 2x = 0 \quad \Leftrightarrow \quad \cos 2x = -\frac{1}{2} \quad \Leftrightarrow \quad 2x = \frac{2\pi}{3}$ or $\frac{4\pi}{3} \quad \Leftrightarrow \quad x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$. $f''(x) = -4\sin 2x$, so $f''\left(\frac{\pi}{3}\right) = -4\sin\frac{2\pi}{3} = -2\sqrt{3} < 0$ and $f''\left(\frac{2\pi}{3}\right) = -4\sin\frac{4\pi}{3} = 2\sqrt{3} > 0$, so $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \approx 1.91$ is a local maximum value and $f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \approx 1.23$ is a local minimum value. Also $f(0) = 0$ and $f(\pi) = \pi$, so $f(0) = 0$ is the absolute minimum value and $f(\pi) = \pi$ is the absolute maximum value.

6. $f(x) = \dfrac{\ln x}{x^2}$, $[1, 3]$. $f'(x) = \dfrac{x^2 \cdot \frac{1}{x} - (\ln x)(2x)}{(x^2)^2} = \dfrac{x - 2x\ln x}{x^4} = \dfrac{1 - 2\ln x}{x^3} = 0 \quad \Leftrightarrow \quad \ln x = \frac{1}{2} \quad \Leftrightarrow$ $x = e^{1/2} = \sqrt{e} \approx 1.65$. $f'(x) > 0$ for $x < \sqrt{e}$ and $f'(x) < 0$ for $x > \sqrt{e}$, so f is increasing on $(1, \sqrt{e}\,)$ and decreasing on $(\sqrt{e}, 3)$. Hence, $f(\sqrt{e}\,) = \dfrac{1}{2e}$ is a local maximum value. $f(1) = 0$ and $f(3) = \dfrac{\ln 3}{9} \approx 0.12$. Since $\dfrac{1}{2e} \approx 0.18$, $f(\sqrt{e}\,) = \dfrac{1}{2e}$ is the absolute maximum value and $f(1) = 0$ is the absolute minimum value.

7. $\displaystyle \lim_{x\to 0} \frac{\tan \pi x}{\ln(1 + x)} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\pi \sec^2 \pi x}{1/(1 + x)} = \frac{\pi \cdot 1^2}{1/1} = \pi$

8. $\displaystyle \lim_{x\to 0} \frac{1 - \cos x}{x^2 + x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\sin x}{2x + 1} = \frac{0}{1} = 0$

9. $\displaystyle \lim_{x\to 0} \frac{e^{4x} - 1 - 4x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0} \frac{4e^{4x} - 4}{2x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{16e^{4x}}{2} = \lim_{x\to 0} 8e^{4x} = 8 \cdot 1 = 8$

10. $\displaystyle \lim_{x\to \infty} \frac{e^{4x} - 1 - 4x}{x^2} \overset{\text{H}}{=} \lim_{x\to \infty} \frac{4e^{4x} - 4}{2x} \overset{\text{H}}{=} \lim_{x\to \infty} \frac{16e^{4x}}{2} = \lim_{x\to \infty} 8e^{4x} = \infty$

11. $\displaystyle \lim_{x\to \infty} x^3 e^{-x} = \lim_{x\to \infty} \frac{x^3}{e^x} \overset{\text{H}}{=} \lim_{x\to \infty} \frac{3x^2}{e^x} \overset{\text{H}}{=} \lim_{x\to \infty} \frac{6x}{e^x} \overset{\text{H}}{=} \lim_{x\to \infty} \frac{6}{e^x} = 0$

12. $\displaystyle \lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} \left(-\tfrac{1}{2}x^2\right) = 0$

13. $$\lim_{x\to 1^+} \left(\frac{x}{x - 1} - \frac{1}{\ln x}\right) = \lim_{x\to 1^+} \left(\frac{x\ln x - x + 1}{(x - 1)\ln x}\right) \overset{\text{H}}{=} \lim_{x\to 1^+} \frac{x \cdot (1/x) + \ln x - 1}{(x - 1) \cdot (1/x) + \ln x}$$
$$= \lim_{x\to 1^+} \frac{\ln x}{1 - 1/x + \ln x} \overset{\text{H}}{=} \lim_{x\to 1^+} \frac{1/x}{1/x^2 + 1/x} = \frac{1}{1 + 1} = \frac{1}{2}$$

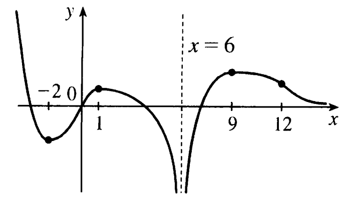
14. $y = (\tan x)^{\cos x} \quad \Rightarrow \quad \ln y = \cos x \ln \tan x$, so
$$\lim_{x\to (\pi/2)^-} \ln y = \lim_{x\to (\pi/2)^-} \frac{\ln \tan x}{\sec x} \overset{\text{H}}{=} \lim_{x\to (\pi/2)^-} \frac{(1/\tan x)\sec^2 x}{\sec x \tan x} = \lim_{x\to (\pi/2)^-} \frac{\sec x}{\tan^2 x}$$
$$= \lim_{x\to (\pi/2)^-} \frac{\cos x}{\sin^2 x} = \frac{0}{1^2} = 0, \text{ so}$$
$$\lim_{x\to (\pi/2)^-} (\tan x)^{\cos x} = \lim_{x\to (\pi/2)^-} e^{\ln y} = e^0 = 1.$$

15. $f(0) = 0$, $f'(-2) = f'(1) = f'(9) = 0$, $\lim\limits_{x\to\infty} f(x) = 0$,

$\lim\limits_{x\to 6} f(x) = -\infty$, $f'(x) < 0$ on $(-\infty, -2)$, $(1, 6)$, and $(9, \infty)$,

$f'(x) > 0$ on $(-2, 1)$ and $(6, 9)$, $f''(x) > 0$ on $(-\infty, 0)$

and $(12, \infty)$, $f''(x) < 0$ on $(0, 6)$ and $(6, 12)$



16. For $0 < x < 1$, $f'(x) = 2x$, so $f(x) = x^2 + C$. Since $f(0) = 0$,

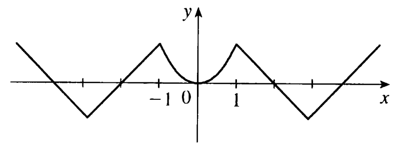
$f(x) = x^2$ on $[0, 1]$. For $1 < x < 3$, $f'(x) = -1$, so

$f(x) = -x + D$. $1 = f(1) = -1 + D \Rightarrow D = 2$, so

$f(x) = 2 - x$. For $x > 3$, $f'(x) = 1$, so $f(x) = x + E$.

$-1 = f(3) = 3 + E \Rightarrow E = -4$, so $f(x) = x - 4$. Since f

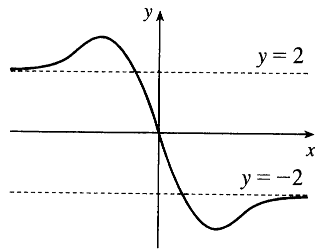
is even, its graph is symmetric about the y-axis.



17. f is odd, $f'(x) < 0$ for $0 < x < 2$, $f'(x) > 0$ for $x > 2$,

$f''(x) > 0$ for $0 < x < 3$, $f''(x) < 0$ for $x > 3$,

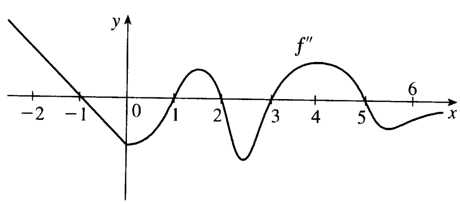
$\lim_{x\to\infty} f(x) = -2$



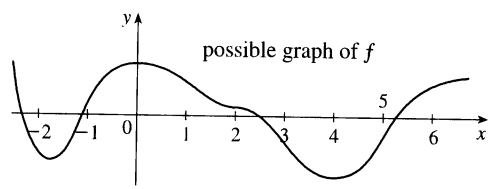
18. (a) Using the Test for Monotonic Functions we know that f is increasing on $(-2, 0)$ and $(4, \infty)$ because $f' > 0$ on $(-2, 0)$ and $(4, \infty)$, and that f is decreasing on $(-\infty, -2)$ and $(0, 4)$ because $f' < 0$ on $(-\infty, -2)$ and $(0, 4)$.

(b) Using the First Derivative Test, we know that f has a local maximum at $x = 0$ because f' changes from positive to negative at $x = 0$, and that f has a local minimum at $x = 4$ because f' changes from negative to positive at $x = 4$.

(c)



(d)

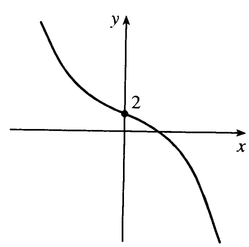


19. $y = f(x) = 2 - 2x - x^3$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$. The x-intercept (approximately 0.770917) can be found using Newton's Method. **C.** No symmetry **D.** No asymptote

E. $f'(x) = -2 - 3x^2 = -(3x^2 + 2) < 0$, so f is decreasing on $\mathbb{R}$.

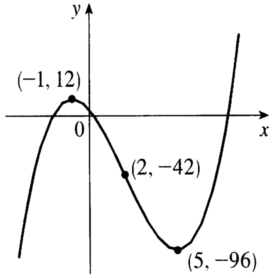
F. No extreme value **G.** $f''(x) = -6x < 0$ on $(0, \infty)$ and $f''(x) > 0$ on $(-\infty, 0)$, so f is CD on $(0, \infty)$ and CU on $(-\infty, 0)$. There is an IP at $(0, 2)$.

H.



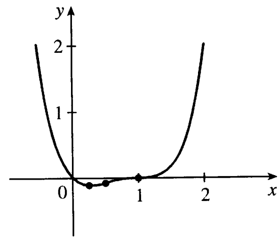
20. $y = f(x) = x^3 - 6x^2 - 15x + 4$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 4$; x-intercepts: $f(x) = 0 \Rightarrow x \approx -2.09, 0.24, 7.85$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5) = 3(x+1)(x-5)$, so f is increasing on $(-\infty, -1)$, decreasing on $(-1, 5)$, and increasing on $(5, \infty)$. **F.** Local maximum value $f(-1) = 12$, local minimum value $f(5) = -96$. **G.** $f''(x) = 6x - 12 = 6(x-2)$, so f is CD on $(-\infty, 2)$ and CU on $(2, \infty)$. There is an IP at $(2, -42)$.

H.



21. $y = f(x) = x^4 - 3x^3 + 3x^2 - x = x(x-1)^3$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow x = 0$ or $x = 1$ **C.** No symmetry **D.** f is a polynomial function and hence, it has no asymptote. **E.** $f'(x) = 4x^3 - 9x^2 + 6x - 1$. Since the sum of the coefficients is 0, 1 is a root of f', so $f'(x) = (x-1)(4x^2 - 5x + 1) = (x-1)^2(4x-1)$. $f'(x) < 0 \Rightarrow x < \frac{1}{4}$, so f is decreasing on $\left(-\infty, \frac{1}{4}\right)$ and f is increasing on $\left(\frac{1}{4}, \infty\right)$. **F.** $f'(x)$ does not change sign at $x = 1$, so there is not a local extremum there. $f\left(\frac{1}{4}\right) = -\frac{27}{256}$ is a local minimum value. **G.** $f''(x) = 12x^2 - 18x + 6 = 6(2x-1)(x-1)$. $f''(x) = 0 \Leftrightarrow x = \frac{1}{2}$ or 1. $f''(x) < 0 \Leftrightarrow \frac{1}{2} < x < 1 \Rightarrow$ f is CD on $\left(\frac{1}{2}, 1\right)$ and CU on $\left(-\infty, \frac{1}{2}\right)$ and $(1, \infty)$. There are inflection points at $\left(\frac{1}{2}, -\frac{1}{16}\right)$ and $(1, 0)$.

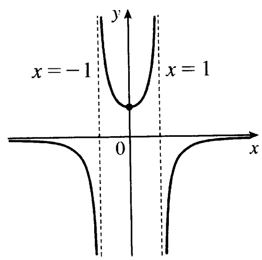
H.



22. $y = f(x) = \dfrac{1}{1-x^2} = \dfrac{1}{(1+x)(1-x)}$ **A.** $D = \{x \mid x \neq \pm 1\}$ **B.** y-intercept: $f(0) = 1$; no x-intercept **C.** $f(-x) = f(x)$, so f is even and the graph of f is symmetric about the y-axis. **D.** Vertical asymptotes: $x = \pm 1$. Horizontal asymptote: $y = 0$ **E.** $y' = \dfrac{2x}{(1-x^2)^2} = 0 \Leftrightarrow x = 0$, so f is decreasing on $(-\infty, -1)$ and $(-1, 0)$, and increasing on $(0, 1)$ and $(1, \infty)$.
F. Local minimum value $f(0) = 1$; no local maximum

G.
$$f''(x) = \frac{(1-x^2)^2 \cdot 2 - 2x \cdot 2(1-x^2)(-2x)}{(1-x^2)^4}$$
$$= \frac{2(1-x^2) + 8x^2}{(1-x^2)^3} = \frac{6x^2 + 2}{(1-x^2)^3} < 0 \Leftrightarrow x^2 > 1,$$
so f is CD on $(-\infty, -1)$ and $(1, \infty)$, and CU on $(-1, 1)$. No IP

H.



23. $y = f(x) = \dfrac{1}{x(x-3)^2}$ **A.** $D = \{x \mid x \neq 0, 3\} = (-\infty, 0) \cup (0, 3) \cup (3, \infty)$ **B.** No intercepts.

C. No symmetry. **D.** $\displaystyle\lim_{x \to \pm\infty} \frac{1}{x(x-3)^2} = 0$, so $y = 0$ is a HA. $\displaystyle\lim_{x \to 0^+} \frac{1}{x(x-3)^2} = \infty$,

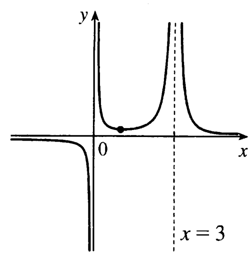
$\displaystyle\lim_{x \to 0^-} \frac{1}{x(x-3)^2} = -\infty$, $\displaystyle\lim_{x \to 3} \frac{1}{x(x-3)^2} = \infty$, so $x = 0$ and $x = 3$ are VA.

E. $f'(x) = -\dfrac{(x-3)^2 + 2x(x-3)}{x^2(x-3)^4} = \dfrac{3(1-x)}{x^2(x-3)^3} \Rightarrow$

$f'(x) > 0 \;\Leftrightarrow\; 1 < x < 3$, so f is increasing on $(1, 3)$ and decreasing on $(-\infty, 0)$, $(0, 1)$, and $(3, \infty)$. **F.** Local minimum value $f(1) = \frac{1}{4}$

G. $f''(x) = \dfrac{6(2x^2 - 4x + 3)}{x^3(x-3)^4}$. Note that $2x^2 - 4x + 3 > 0$ for all x since it has negative discriminant. So $f''(x) > 0 \;\Leftrightarrow\; x > 0 \;\Rightarrow\; f$ is CU on $(0, 3)$ and $(3, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



24. $y = f(x) = \dfrac{1}{x} + \dfrac{1}{x+1} = \dfrac{2x+1}{x(x+1)}$ **A.** $D = \{x \mid x \neq 0, -1\}$ **B.** No y-intercept, x-intercept $= -\frac{1}{2}$

C. No symmetry **D.** $\lim\limits_{x\to\pm\infty} f(x) = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to 0^+} \dfrac{2x+1}{x(x+1)} = \infty$, $\lim\limits_{x\to 0^-} \dfrac{2x+1}{x(x+1)} = -\infty$,

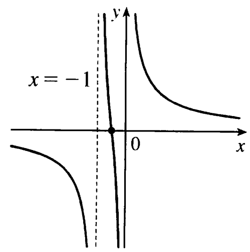
$\lim\limits_{x\to -1^+} \dfrac{2x+1}{x(x+1)} = \infty$, $\lim\limits_{x\to -1^-} \dfrac{2x+1}{x(x+1)} = -\infty$, so $x = 0$, $x = -1$ are VA.

E. $f'(x) = -\dfrac{1}{x^2} - \dfrac{1}{(x+1)^2} < 0$, so f is decreasing on $(-\infty, -1)$, $(-1, 0)$ and $(0, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{2}{x^3} + \dfrac{2}{(x+1)^3} = \dfrac{2(2x+1)(x^2+x+1)}{x^3(x+1)^3}$.

$f''(x) > 0 \;\Leftrightarrow\; x > 0$ or $-1 < x < -\frac{1}{2}$, so f is CU on $(0, \infty)$ and $\left(-1, -\frac{1}{2}\right)$ and CD on $(-\infty, -1)$ and $\left(-\frac{1}{2}, 0\right)$. IP at $\left(-\frac{1}{2}, 0\right)$

H.



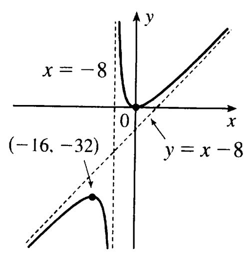
25. $y = f(x) = \dfrac{x^2}{x+8} = x - 8 + \dfrac{64}{x+8}$ **A.** $D = \{x \mid x \neq -8\}$ **B.** Intercepts are 0 **C.** No symmetry

D. $\lim\limits_{x\to\infty} \dfrac{x^2}{x+8} = \infty$, but $f(x) - (x - 8) = \dfrac{64}{x+8} \to 0$ as $x \to \infty$, so $y = x - 8$ is a slant asymptote.

$\lim\limits_{x\to -8^+} \dfrac{x^2}{x+8} = \infty$ and $\lim\limits_{x\to -8^-} \dfrac{x^2}{x+8} = -\infty$, so $x = -8$ is a VA.

E. $f'(x) = 1 - \dfrac{64}{(x+8)^2} = \dfrac{x(x+16)}{(x+8)^2} > 0 \;\Leftrightarrow\; x > 0$ or $x < -16$, so f is increasing on $(-\infty, -16)$ and $(0, \infty)$ and decreasing on $(-16, -8)$ and $(-8, 0)$. **F.** Local maximum value $f(-16) = -32$, local minimum value $f(0) = 0$ **G.** $f''(x) = 128/(x+8)^3 > 0 \;\Leftrightarrow\; x > -8$, so f is CU on $(-8, \infty)$ and CD on $(-\infty, -8)$. No IP

H.

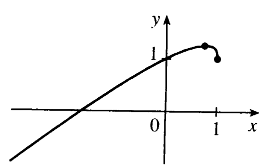


26. $y = f(x) = x + \sqrt{1-x}$ **A.** $D = \{x \mid x \leq 1\} = (-\infty, 1]$ **B.** y-intercept $= 1$; x-intercepts occur when $x + \sqrt{1-x} = 0 \;\Rightarrow\; \sqrt{1-x} = -x \;\Rightarrow\; 1 - x = x^2 \;\Rightarrow\; x^2 + x - 1 = 0 \;\Rightarrow\; x = \frac{-1\pm\sqrt{5}}{2}$, but the larger root is extraneous, so the only x-intercept is $\frac{-1-\sqrt{5}}{2}$. **C.** No symmetry **D.** No asymptote

E. $f'(x) = 1 - 1/\left(2\sqrt{1-x}\right) = 0 \;\Leftrightarrow\; 2\sqrt{1-x} = 1 \;\Leftrightarrow\; 1 - x = \frac{1}{4} \;\Leftrightarrow\; x = \frac{3}{4}$ and $f'(x) > 0 \;\Leftrightarrow\; x < \frac{3}{4}$, so f is increasing on $\left(-\infty, \frac{3}{4}\right)$, decreasing on $\left(\frac{3}{4}, 1\right)$. **F.** Local maximum value $f\left(\frac{3}{4}\right) = \frac{5}{4}$

G. $f''(x) = -\dfrac{1}{4(1-x)^{3/2}} < 0 \;\Leftrightarrow\; x < 1$, so f is CD on $(-\infty, 1)$. No IP

H.

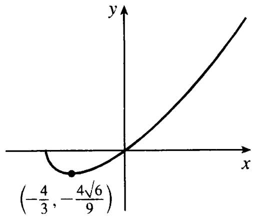


27. $y = f(x) = x\sqrt{2+x}$ **A.** $D = [-2, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: -2 and 0 **C.** No symmetry **D.** No asymptote **E.** $f'(x) = \dfrac{x}{2\sqrt{2+x}} + \sqrt{2+x} = \dfrac{1}{2\sqrt{2+x}}[x + 2(2+x)] = \dfrac{3x+4}{2\sqrt{2+x}} = 0$ when $x = -\frac{4}{3}$, so f is decreasing on $\left(-2, -\frac{4}{3}\right)$ and increasing on $\left(-\frac{4}{3}, \infty\right)$. **F.** Local minimum value $f\left(-\frac{4}{3}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}} = -\frac{4\sqrt{6}}{9} \approx -1.09$, no local maximum

G.
$$f''(x) = \frac{2\sqrt{2+x}\cdot 3 - (3x+4)\dfrac{1}{\sqrt{2+x}}}{4(2+x)} = \frac{6(2+x)-(3x+4)}{4(2+x)^{3/2}} = \frac{3x+8}{4(2+x)^{3/2}}$$

$f''(x) > 0$ for $x > -2$, so f is CU on $(-2, \infty)$. No IP

H.



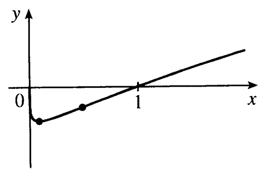
28. $y = f(x) = \sqrt{x} - \sqrt[3]{x}$ **A.** $D = [0, \infty)$ **B.** y-intercept 0; x-intercepts 0, 1 **C.** No symmetry **D.** $\lim\limits_{x\to\infty}\left(x^{1/2} - x^{1/3}\right) = \lim\limits_{x\to\infty}\left[x^{1/3}\left(x^{1/6} - 1\right)\right] = \infty$, no asymptote

E. $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{3}x^{-2/3} = \dfrac{3x^{1/6} - 2}{6x^{2/3}} > 0 \quad\Leftrightarrow\quad 3x^{1/6} > 2 \quad\Leftrightarrow\quad x > \left(\frac{2}{3}\right)^6$, so f is increasing on $\left(\left(\frac{2}{3}\right)^6, \infty\right)$ and decreasing on $\left(0, \left(\frac{2}{3}\right)^6\right)$. **F.** $f\left(\left(\frac{2}{3}\right)^6\right) = -\frac{4}{27}$ is a local minimum value.

G. $f''(x) = -\frac{1}{4}x^{-3/2} + \frac{2}{9}x^{-5/3} = \dfrac{8 - 9x^{1/6}}{36x^{5/3}} > 0 \quad\Leftrightarrow\quad x^{1/6} < \frac{8}{9}$ $\quad\Leftrightarrow\quad x < \left(\frac{8}{9}\right)^6$, so f is CU on $\left(0, \left(\frac{8}{9}\right)^6\right)$ and CD on $\left(\left(\frac{8}{9}\right)^6, \infty\right)$.

IP at $\left(\left(\frac{8}{9}\right)^6, -\frac{64}{729}\right)$

H.



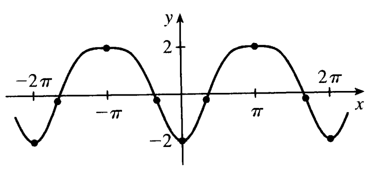
29. $y = f(x) = \sin^2 x - 2\cos x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = -2$ **C.** $f(-x) = f(x)$, so f is symmetric with respect to the y-axis. f has period 2π. **D.** No asymptote

E. $y' = 2\sin x\cos x + 2\sin x = 2\sin x(\cos x + 1)$. $y' = 0 \quad\Leftrightarrow\quad \sin x = 0$ or $\cos x = -1 \quad\Leftrightarrow\quad x = n\pi$ or $x = (2n+1)\pi$. $y' > 0$ when $\sin x > 0$, since $\cos x + 1 \geq 0$ for all x. Therefore, $y' > 0$ (and so f is increasing) on $(2n\pi, (2n+1)\pi)$; $y' < 0$ (and so f is decreasing) on $((2n-1)\pi, 2n\pi)$. **F.** Local maximum values are $f((2n+1)\pi) = 2$; local minimum values are $f(2n\pi) = -2$. **G.** $y' = \sin 2x + 2\sin x \quad\Rightarrow$

$$y'' = 2\cos 2x + 2\cos x = 2\left(2\cos^2 x - 1\right) + 2\cos x = 4\cos^2 x + 2\cos x - 2$$
$$= 2\left(2\cos^2 x + \cos x - 1\right) = 2(2\cos x - 1)(\cos x + 1)$$

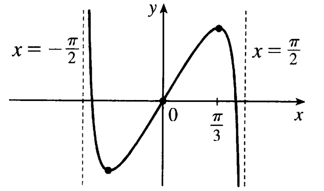
$y'' = 0 \quad\Leftrightarrow\quad \cos x = \frac{1}{2}$ or $-1 \quad\Leftrightarrow\quad x = 2n\pi \pm \frac{\pi}{3}$ or $x = (2n+1)\pi$. $y'' > 0$ (and so f is CU) on $\left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right)$; $y'' \leq 0$ (and so f is CD) on $\left(2n\pi + \frac{\pi}{3}, 2n\pi + \frac{5\pi}{3}\right)$. There are inflection points at $\left(2n\pi \pm \frac{\pi}{3}, -\frac{1}{4}\right)$.

H.



30. $y = f(x) = 4x - \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. **B.** y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\lim\limits_{x\to\pi/2^-}(4x - \tan x) = -\infty$, $\lim\limits_{x\to-\pi/2^+}(4x - \tan x) = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA. **E.** $f'(x) = 4 - \sec^2 x > 0 \iff \sec x < 2 \iff \cos x > \frac{1}{2} \iff -\frac{\pi}{3} < x < \frac{\pi}{3}$, so f is increasing on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ and decreasing on $\left(-\frac{\pi}{2}, -\frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$. **F.** $f\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3}$ is a local maximum value, $f\left(-\frac{\pi}{3}\right) = \sqrt{3} - \frac{4\pi}{3}$ is a local minimum value. **G.** $f''(x) = -2\sec^2 x \tan x > 0 \iff \tan x < 0 \iff -\frac{\pi}{2} < x < 0$, so f is CU on $\left(-\frac{\pi}{2}, 0\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$. IP at $(0, 0)$

H.



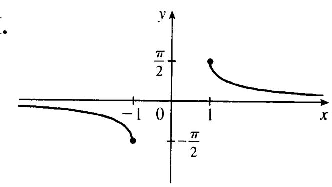
31. $y = f(x) = \sin^{-1}(1/x)$ **A.** $D = \{x \mid -1 \le 1/x \le 1\} = (-\infty, -1] \cup [1, \infty)$. **B.** No intercept **C.** $f(-x) = -f(x)$, symmetric about the origin **D.** $\lim\limits_{x\to\pm\infty} \sin^{-1}(1/x) = \sin^{-1}(0) = 0$, so $y = 0$ is a HA.

E. $f'(x) = \dfrac{1}{\sqrt{1 - (1/x)^2}}\left(-\dfrac{1}{x^2}\right) = \dfrac{-1}{\sqrt{x^4 - x^2}} < 0$, so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

F. No local extreme value, but $f(1) = \frac{\pi}{2}$ is the absolute maximum value and $f(-1) = -\frac{\pi}{2}$ is the absolute minimum value.

G. $f''(x) = \dfrac{4x^3 - 2x}{2(x^4 - x^2)^{3/2}} = \dfrac{x(2x^2 - 1)}{(x^4 - x^2)^{3/2}} > 0$ for $x > 1$ and $f''(x) < 0$ for $x < -1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, -1)$. No IP

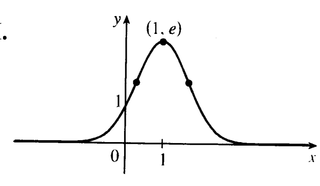
H.



32. $y = f(x) = e^{2x - x^2}$ **A.** $D = \mathbb{R}$ **B.** y-intercept 1; no x-intercept **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty} e^{2x-x^2} = 0$, so $y = 0$ is a HA. **E.** $y = f(x) = e^{2x-x^2} \Rightarrow f'(x) = 2(1 - x)e^{2x-x^2} > 0 \iff x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. **F.** $f(1) = e$ is a local and absolute maximum value.

G. $f''(x) = 2(2x^2 - 4x + 1)e^{2x-x^2} = 0 \iff x = 1 \pm \frac{\sqrt{2}}{2}$. $f''(x) > 0 \iff x < 1 - \frac{\sqrt{2}}{2}$ or $x > 1 + \frac{\sqrt{2}}{2}$, so f is CU on $\left(-\infty, 1 - \frac{\sqrt{2}}{2}\right)$ and $\left(1 + \frac{\sqrt{2}}{2}, \infty\right)$, and CD on $\left(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}\right)$. IP at $\left(1 \pm \frac{\sqrt{2}}{2}, \sqrt{e}\right)$

H.

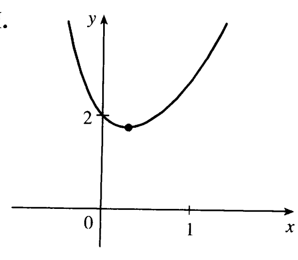


33. $y = f(x) = e^x + e^{-3x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept 2; no x-intercept **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty}\left(e^x + e^{-3x}\right) = \infty$, no asymptote **E.** $y = f(x) = e^x + e^{-3x} \Rightarrow$ $f'(x) = e^x - 3e^{-3x} = e^{-3x}\left(e^{4x} - 3\right) > 0 \iff e^{4x} > 3 \iff 4x > \ln 3 \iff x > \frac{1}{4}\ln 3 \approx 0.27$, so f is increasing on $\left(\frac{1}{4}\ln 3, \infty\right)$ and decreasing on $\left(-\infty, \frac{1}{4}\ln 3\right)$.

F. Absolute minimum value $f\left(\frac{1}{4}\ln 3\right) = 3^{1/4} + 3^{-3/4} \approx 1.75$.

G. $f''(x) = e^x + 9e^{-3x} > 0$, so f is CU on $(-\infty, \infty)$. No IP

H.

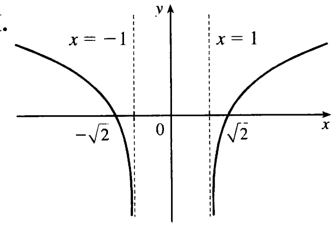


34. $y = f(x) = \ln(x^2 - 1)$ **A.** $D = (-\infty, -1) \cup (1, \infty)$ **B.** No y-intercept; x-intercepts $\pm\sqrt{2}$ **C.** Symmetric about the y-axis **D.** $\lim\limits_{x\to\pm\infty} \ln(x^2 - 1) = \infty$, $\lim\limits_{x\to 1^+} \ln(x^2 - 1) = -\infty$, $\lim\limits_{x\to -1^-} \ln(x^2 - 1) = -\infty$, so $x = 1$ and $x = -1$ are VA. **E.** $y = f(x) = \ln(x^2 - 1) \;\Rightarrow\; f'(x) = \dfrac{2x}{x^2 - 1} > 0$ for $x > 1$ and $f'(x) < 0$ for $x < -1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, -1)$. Note that the domain of f is $|x| > 1$. **F.** No extreme value

G. $f''(x) = -2\dfrac{x^2 + 1}{(x^2 - 1)^2} < 0$, so f is CD on $(-\infty, -1)$ and $(1, \infty)$.

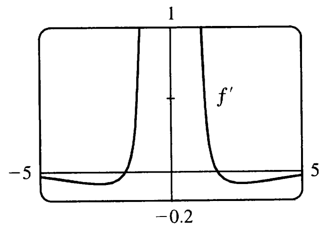
No IP

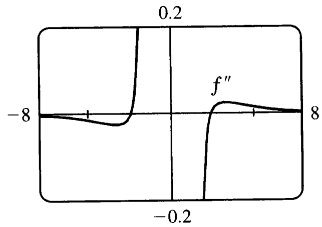
H.

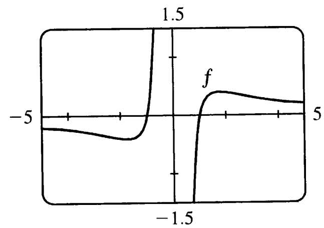


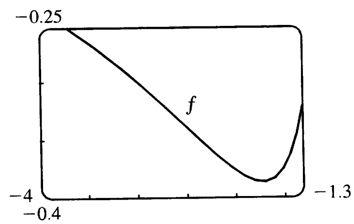
35. $f(x) = \dfrac{x^2 - 1}{x^3} \;\Rightarrow\; f'(x) = \dfrac{x^3(2x) - (x^2 - 1)3x^2}{x^6} = \dfrac{3 - x^2}{x^4} \;\Rightarrow$

$f''(x) = \dfrac{x^4(-2x) - (3 - x^2)4x^3}{x^8} = \dfrac{2x^2 - 12}{x^5}$









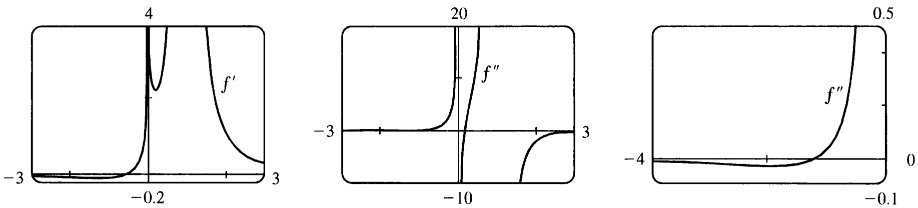
Estimates: From the graphs of f' and f'', it appears that f is increasing on $(-1.73, 0)$ and $(0, 1.73)$ and decreasing on $(-\infty, -1.73)$ and $(1.73, \infty)$; f has a local maximum of about $f(1.73) = 0.38$ and a local minimum of about $f(-1.7) = -0.38$; f is CU on $(-2.45, 0)$ and $(2.45, \infty)$, and CD on $(-\infty, -2.45)$ and $(0, 2.45)$; and f has inflection points at about $(-2.45, -0.34)$ and $(2.45, 0.34)$.

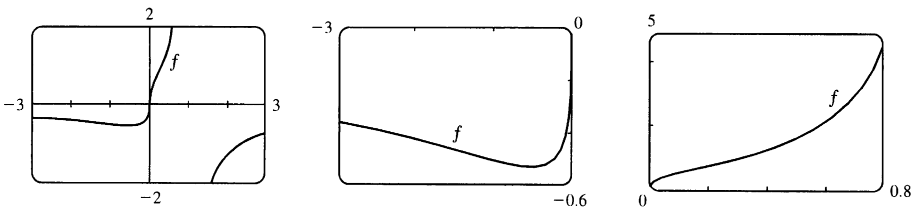
Exact: Now $f'(x) = \dfrac{3 - x^2}{x^4}$ is positive for $0 < x^2 < 3$, that is, f is increasing on $(-\sqrt{3}, 0)$ and $(0, \sqrt{3})$; and $f'(x)$ is negative (and so f is decreasing) on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$. $f'(x) = 0$ when $x = \pm\sqrt{3}$. f' goes from positive to negative at $x = \sqrt{3}$, so f has a local maximum of $f(\sqrt{3}) = \frac{(\sqrt{3})^2 - 1}{(\sqrt{3})^3} = \frac{2\sqrt{3}}{9}$; and since f is odd, we know that maxima on the interval $(0, \infty)$ correspond to minima on $(-\infty, 0)$, so f has a local minimum of $f(-\sqrt{3}) = -\frac{2\sqrt{3}}{9}$. Also, $f''(x) = \dfrac{2x^2 - 12}{x^5}$ is positive (so f is CU) on $(-\sqrt{6}, 0)$ and $(\sqrt{6}, \infty)$, and negative (so f is CD) on $(-\infty, -\sqrt{6})$ and $(0, \sqrt{6})$. There are IP at $\left(\sqrt{6}, \frac{5\sqrt{6}}{36}\right)$ and $\left(-\sqrt{6}, -\frac{5\sqrt{6}}{36}\right)$.

36. $f(x) = \dfrac{\sqrt[3]{x}}{1-x} = x^{1/3}(1-x)^{-1} \quad \Rightarrow$

$$f'(x) = x^{1/3}(-1)(1-x)^{-2}(-1) + (1-x)^{-1}\left(\tfrac{1}{3}\right)x^{-2/3} = \frac{x^{-2/3}}{3}\,\frac{1+2x}{(x-1)^2} \quad \Rightarrow$$

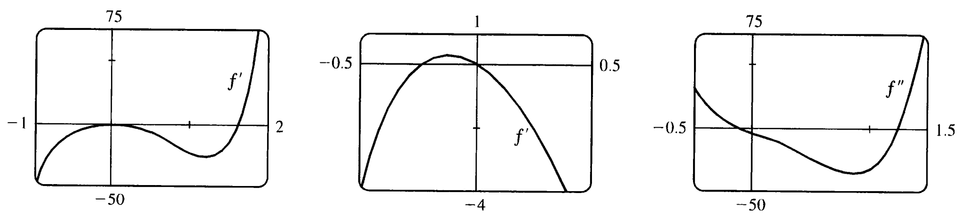
$$f''(x) = \frac{x^{-2/3}}{3}\,\frac{(x-1)^2(2)-(1+2x)(2)(x-1)}{(x-1)^4} + \frac{1+2x}{(x-1)^2}\left(\frac{-2x^{-5/3}}{9}\right) = -\frac{2x^{-5/3}}{9}\,\frac{5x^2+5x-1}{(x-1)^3}$$

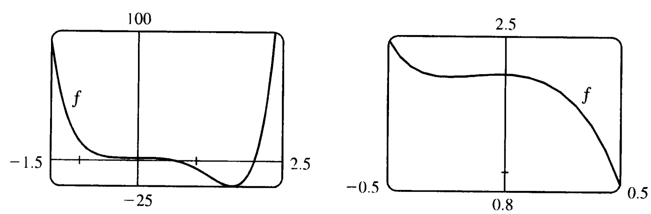




From the graphs, it appears that f is increasing on $(-0.50, 1)$ and $(1, \infty)$, with a vertical asymptote at $x = 1$, and decreasing on $(-\infty, -0.50)$; f has no local maximum, but a local minimum of about $f(-0.50) = -0.53$; f is CU on $(-1.17, 0)$ and $(0.17, 1)$ and CD on $(-\infty, -1.17)$, $(0, 0.17)$ and $(1, \infty)$; and f has inflection points at about $(-1.17, -0.49)$, $(0, 0)$ and $(0.17, 0.67)$. Note also that $\lim\limits_{x\to\pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote.

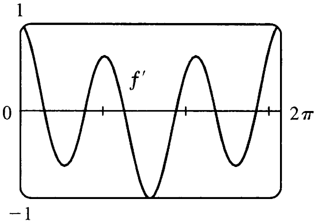
37. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2 \quad \Rightarrow \quad f'(x) = 18x^5 - 25x^4 + 4x^3 - 15x^2 - 4x \quad \Rightarrow$
$f''(x) = 90x^4 - 100x^3 + 12x^2 - 30x - 4$

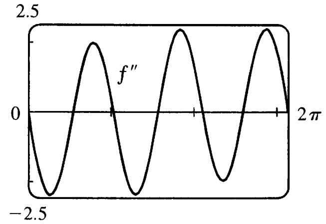


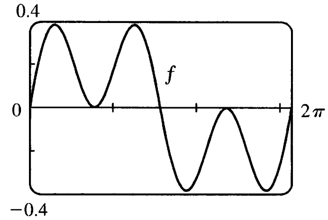


From the graphs of f' and f'', it appears that f is increasing on $(-0.23, 0)$ and $(1.62, \infty)$ and decreasing on $(-\infty, -0.23)$ and $(0, 1.62)$; f has a local maximum of about $f(0) = 2$ and local minima of about $f(-0.23) = 1.96$ and $f(1.62) = -19.2$; f is CU on $(-\infty, -0.12)$ and $(1.24, \infty)$ and CD on $(-0.12, 1.24)$; and f has inflection points at about $(-0.12, 1.98)$ and $(1.24, -12.1)$.

38. $f(x) = \sin x \cos^2 x \quad \Rightarrow \quad f'(x) = \cos^3 x - 2\sin^2 x \cos x \quad \Rightarrow \quad f''(x) = -7\sin x \cos^2 x + 2\sin^3 x$

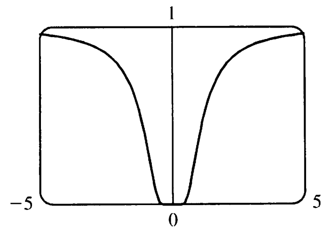






From the graphs of f' and f'', it appears that f is increasing on $(0, 0.62)$, $(1.57, 2.53)$, $(3.76, 4.71)$ and $(5.67, 2\pi)$ and decreasing on $(0.62, 1.57)$, $(2.53, 3.76)$ and $(4.71, 5.67)$; f has local maxima of about $f(0.62) = f(2.53) = 0.38$ and $f(4.71) = 0$ and local minima of about $f(1.57) = 0$ and $f(3.76) = f(5.67) = -0.38$; f is CU on $(1.08, 2.06)$, $(3.14, 4.22)$ and $(5.20, 2\pi)$ and CD on $(0, 1.08)$, $(2.06, 3.14)$ and $(4.22, 5.20)$; and f has inflection points at about $(0, 0)$, $(1.08, 0.20)$, $(2.06, 0.20)$, $(3.14, 0)$, $(4.22, -0.20)$, $(5.20, -0.20)$ and $(2\pi, 0)$.

39.

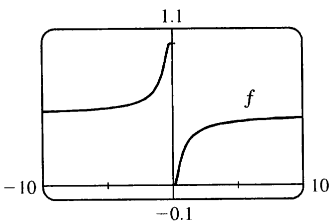


From the graph, we estimate the points of inflection to be about $(\pm 0.82, 0.22)$. $f(x) = e^{-1/x^2} \quad \Rightarrow \quad f'(x) = 2x^{-3}e^{-1/x^2} \quad \Rightarrow$

$$f''(x) = 2\left[x^{-3}(2x^{-3})e^{-1/x^2} + e^{-1/x^2}(-3x^{-4})\right]$$
$$= 2x^{-6}e^{-1/x^2}\,(2 - 3x^2).$$

This is 0 when $2 - 3x^2 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt{\frac{2}{3}}$, so the inflection points are $\left(\pm\sqrt{\frac{2}{3}}, e^{-3/2}\right)$.

40. (a)

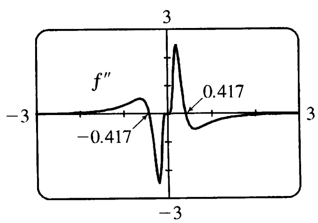


(b) $f(x) = \dfrac{1}{1 + e^{1/x}}$. $\displaystyle\lim_{x\to\infty} f(x) = \frac{1}{1+1} = \frac{1}{2}$,

$$\lim_{x\to-\infty} f(x) = \frac{1}{1+1} = \frac{1}{2},$$
$$\lim_{x\to0^+} f(x) = \frac{1}{1+\infty} = 0,$$
$$\lim_{x\to0^-} f(x) = \frac{1}{1+0} = 1$$

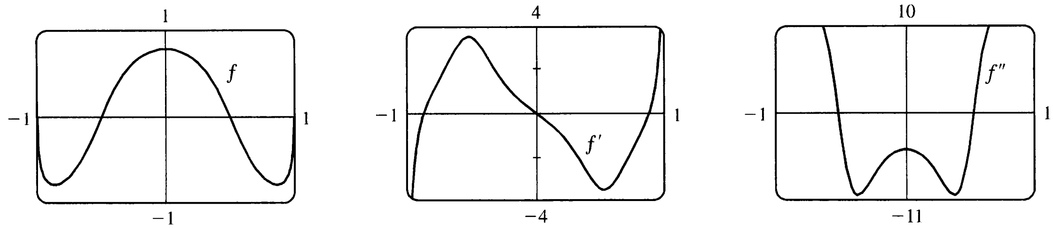
(c) From the graph of f, estimates for the IP are $(-0.4, 0.9)$ and $(0.4, 0.08)$.

(d) $f''(x) = -\dfrac{e^{1/x}\left[e^{1/x}(2x-1) + 2x + 1\right]}{x^4\left(e^{1/x}+1\right)^3}$



(e) From the graph, we see that f'' changes sign at $x = \pm 0.417$ ($x = 0$ is not in the domain of f). IP are approximately $(0.417, 0.083)$ and $(-0.417, 0.917)$.

41. $f(x) = \arctan(\cos(3\arcsin x))$. We use a CAS to compute f' and f'', and to graph f, f', and f'':

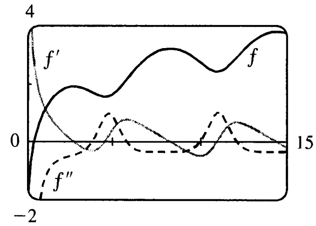


From the graph of f', it appears that the only maximum occurs at $x = 0$ and there are minima at $x = \pm 0.87$. From the graph of f'', it appears that there are inflection points at $x = \pm 0.52$.

42. $f(x) = \ln(2x + x\sin x)$. We use the CAS to calculate

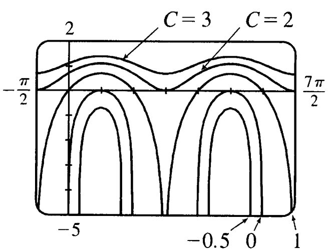
$$f'(x) = \frac{2 + \sin x + x\cos x}{2x + x\sin x} \text{ and}$$

$$f''(x) = \frac{2x^2\sin x + 4\sin x - \cos^2 x + x^2 + 5}{x^2(\cos^2 x - 4\sin x - 5)}.$$



From the graphs, it seems that $f' > 0$ (and so f is increasing) on approximately the intervals $(0, 2.7)$, $(4.5, 8.2)$ and $(10.9, 14.3)$. It seems that f'' changes sign (indicating inflection points) at $x \approx 3.8$, 5.7, 10.0 and 12.0. Looking back at the graph of f, this implies that the inflection points have approximate coordinates $(3.8, 1.7)$, $(5.7, 2.1)$, $(10.0, 2.7)$, and $(12.0, 2.9)$.

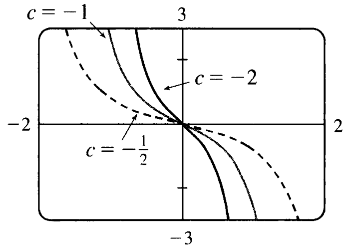
43. The family of functions $f(x) = \ln(\sin x + C)$ all have the same period and all have maximum values at $x = \frac{\pi}{2} + 2\pi n$. Since the domain of ln is $(0, \infty)$, f has a graph only if $\sin x + C > 0$ somewhere. Since $-1 \le \sin x \le 1$, this happens if $C > -1$, that is, f has no graph if $C \le -1$. Similarly, if $C > 1$, then $\sin x + C > 0$ and f is continuous on $(-\infty, \infty)$. As C increases, the graph of f is shifted vertically upward and flattens out.

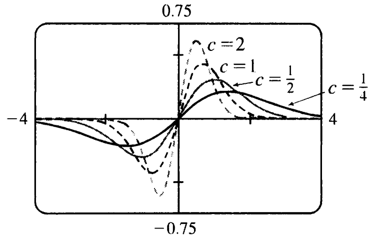


If $-1 < C \le 1$, f is defined where $\sin x + C > 0 \iff \sin x > -C \iff \sin^{-1}(-C) < x < \pi - \sin^{-1}(-C)$. Since the period is 2π, the domain of f is $\left(2n\pi + \sin^{-1}(-C), (2n+1)\pi - \sin^{-1}(-C)\right)$, n an integer.

44. We exclude the case $c = 0$, since in that case $f(x) = 0$ for all x. To find the maxima and minima, we differentiate: $f(x) = cxe^{-cx^2} \Rightarrow f'(x) = c\left[xe^{-cx^2}(-2cx) + e^{-cx^2}(1)\right] = ce^{-cx^2}\left(-2cx^2 + 1\right)$. This is 0 where $-2cx^2 + 1 = 0 \iff x = \pm 1/\sqrt{2c}$. So if $c > 0$, there are two maxima or minima, whose x-coordinates approach 0 as c increases. The negative root gives a minimum and the positive root gives a maximum, by the First Derivative Test. By substituting back into the equation, we see that $f\left(\pm 1/\sqrt{2c}\right) = c\left(\pm 1/\sqrt{2c}\right)e^{-c\left(\pm 1/\sqrt{2c}\right)^2} = \pm\sqrt{c/2e}$. So as c increases, the extreme points become more pronounced. Note that if $c > 0$, then $\lim\limits_{x \to \pm\infty} f(x) = 0$. If $c < 0$, then there are no extreme values, and $\lim\limits_{x \to \pm\infty} f(x) = \mp\infty$.

To find the points of inflection, we differentiate again: $f'(x) = ce^{-cx^2}(-2cx^2 + 1) \Rightarrow$
$f''(x) = c\left[e^{-cx^2}(-4cx) + (-2cx^2+1)\left(-2cxe^{-cx^2}\right)\right] = -2c^2xe^{-cx^2}(3 - 2cx^2)$. This is 0 at $x = 0$ and where $3 - 2cx^2 = 0 \Leftrightarrow x = \pm\sqrt{3/(2c)} \Rightarrow$ IP at $\left(\pm\sqrt{3/(2c)}, \pm\sqrt{3c/2}\,e^{-3/2}\right)$. If $c > 0$ there are three inflection points, and as c increases, the x-coordinates of the nonzero inflection points approach 0. If $c < 0$, there is only one inflection point, the origin.





45. $f(x) = x^{101} + x^{51} + x - 1 = 0$. Since f is continuous and $f(0) = -1$ and $f(1) = 2$, the equation has at least one root in $(0, 1)$, by the Intermediate Value Theorem. Suppose the equation has two roots, a and b, with $a < b$. Then $f(a) = 0 = f(b)$, so by the Mean Value Theorem, there is a number x in (a, b) such that
$f'(x) = \dfrac{f(b) - f(a)}{b - a} = \dfrac{0}{b - a} = 0$, so f' has a root in (a, b). But this is impossible since
$f'(x) = 101x^{100} + 51x^{50} + 1 \geq 1$ for all x.

46. By the Mean Value Theorem, $f'(c) = \dfrac{f(4) - f(0)}{4 - 0} \Leftrightarrow 4f'(c) = f(4) - 1$ for some c with $0 < c < 4$. Since $2 \leq f'(c) \leq 5$, we have $4(2) \leq 4f'(c) \leq 4(5) \Leftrightarrow 4(2) \leq f(4) - 1 \leq 4(5) \Leftrightarrow 8 \leq f(4) - 1 \leq 20 \Leftrightarrow$
$9 \leq f(4) \leq 21$.

47. Since f is continuous on $[32, 33]$ and differentiable on $(32, 33)$, then by the Mean Value Theorem there exists a number c in $(32, 33)$ such that $f'(c) = \frac{1}{5}c^{-4/5} = \dfrac{\sqrt[5]{33} - \sqrt[5]{32}}{33 - 32} = \sqrt[5]{33} - 2$, but $\frac{1}{5}c^{-4/5} > 0 \Rightarrow \sqrt[5]{33} - 2 > 0$
$\Rightarrow \sqrt[5]{33} > 2$. Also f' is decreasing, so that $f'(c) < f'(32) = \frac{1}{5}(32)^{-4/5} = 0.0125 \Rightarrow$
$0.0125 > f'(c) = \sqrt[5]{33} - 2 \Rightarrow \sqrt[5]{33} < 2.0125$. Therefore, $2 < \sqrt[5]{33} < 2.0125$.

48. For $(1, 6)$ to be on the curve $y = x^3 + ax^2 + bx + 1$, we have that $6 = 1 + a + b + 1 \Rightarrow b = 4 - a$. Now $y' = 3x^2 + 2ax + b$ and $y'' = 6x + 2a$. Also, for $(1, 6)$ to be an inflection point it must be true that
$y''(1) = 6(1) + 2a = 0 \Rightarrow a = -3 \Rightarrow b = 4 - (-3) = 7$. Note that with $a = -3$, we have
$y'' = 6x - 6 = 6(x - 1)$, so y'' changes sign at $x = 1$, proving that $(1, 6)$ is a point of inflection. [This does not follow from the fact that $y''(1) = 0$.]

49. (a) $g(x) = f(x^2) \Rightarrow g'(x) = 2xf'(x^2)$ by the Chain Rule. Since $f'(x) > 0$ for all $x \neq 0$, we must have $f'(x^2) > 0$ for $x \neq 0$, so $g'(x) = 0 \Leftrightarrow x = 0$. Now $g'(x)$ changes sign (from negative to positive) at

$x = 0$, since one of its factors, $f'(x^2)$, is positive for all x, and its other factor, $2x$, changes from negative to positive at this point, so by the First Derivative Test, f has a local and absolute minimum at $x = 0$.

(b) $g'(x) = 2xf'(x^2) \Rightarrow g''(x) = 2\left[xf''(x^2)(2x) + f'(x^2)\right] = 4x^2f''(x^2) + 2f'(x^2)$ by the Product Rule and the Chain Rule. But $x^2 > 0$ for all $x \neq 0$, $f''(x^2) > 0$ (since f is CU for $x > 0$), and $f'(x^2) > 0$ for all $x \neq 0$, so since all of its factors are positive, $g''(x) > 0$ for $x \neq 0$. Whether $g''(0)$ is positive or 0 doesn't matter (since the sign of g'' does not change there); g is concave upward on $\mathbb{R}$.

50. Call the two integers x and y. Then $x + 4y = 1000$, so $x = 1000 - 4y$. Their product is $P = xy = (1000 - 4y)y$, so our problem is to maximize the function $P(y) = 1000y - 4y^2$, where $0 < y < 250$ and y is an integer. $P'(y) = 1000 - 8y$, so $P'(y) = 0 \Leftrightarrow y = 125$. $P''(y) = -8 < 0$, so $P(125) = 62{,}500$ is an absolute maximum. Since the optimal y turned out to be an integer, we have found the desired pair of numbers, namely $x = 1000 - 4(125) = 500$ and $y = 125$.

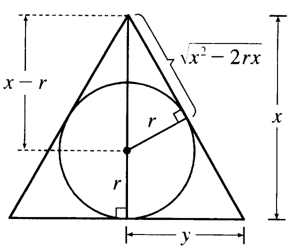
51. If $B = 0$, the line is vertical and the distance from $x = -\dfrac{C}{A}$ to (x_1, y_1) is $\left|x_1 + \dfrac{C}{A}\right| = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$, so assume $B \neq 0$. The square of the distance from (x_1, y_1) to the line is $f(x) = (x - x_1)^2 + (y - y_1)^2$ where $Ax + By + C = 0$, so we minimize $f(x) = (x - x_1)^2 + \left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)^2 \Rightarrow$

$f'(x) = 2(x - x_1) + 2\left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)\left(-\dfrac{A}{B}\right)$. $f'(x) = 0 \Rightarrow x = \dfrac{B^2x_1 - ABy_1 - AC}{A^2 + B^2}$ and this gives a minimum since $f''(x) = 2\left(1 + \dfrac{A^2}{B^2}\right) > 0$. Substituting this value of x into $f(x)$ and simplifying gives

$f(x) = \dfrac{(Ax_1 + By_1 + C)^2}{A^2 + B^2}$, so the minimum distance is $\sqrt{f(x)} = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

52. On the hyperbola $xy = 8$, if $d(x)$ is the distance from the point $(x, y) = (x, 8/x)$ to the point $(3, 0)$, then $[d(x)]^2 = (x - 3)^2 + 64/x^2 = f(x)$. $f'(x) = 2(x - 3) - 128/x^3 = 0 \Rightarrow x^4 - 3x^3 - 64 = 0 \Rightarrow (x - 4)(x^3 + x^2 + 4x + 16) = 0 \Rightarrow x = 4$ since the solution must have $x > 0$. Then $y = \frac{8}{4} = 2$, so the point is $(4, 2)$.

53.



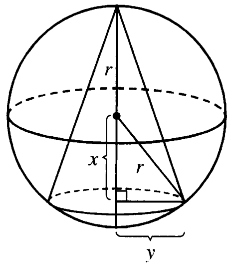
By similar triangles, $\dfrac{y}{x} = \dfrac{r}{\sqrt{x^2 - 2rx}}$, so the area of the triangle is

$$A(x) = \tfrac{1}{2}(2y)x = xy = \frac{rx^2}{\sqrt{x^2 - 2rx}} \Rightarrow$$

$$A'(x) = \frac{2rx\sqrt{x^2 - 2rx} - rx^2(x - r)/\sqrt{x^2 - 2rx}}{x^2 - 2rx} = \frac{rx^2(x - 3r)}{(x^2 - 2rx)^{3/2}} = 0 \text{ when } x = 3r.$$

$A'(x) < 0$ when $2r < x < 3r$, $A'(x) > 0$ when $x > 3r$. So $x = 3r$ gives a minimum and $A(3r) = r(9r^2)/(\sqrt{3}\,r) = 3\sqrt{3}\,r^2$.

54.



The volume of the cone is

$V = \frac{1}{3}\pi y^2(r+x) = \frac{1}{3}\pi(r^2 - x^2)(r+x),\ -r \le x \le r.$

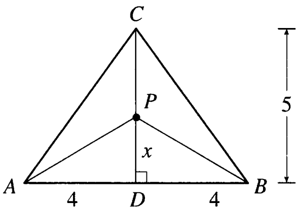
$V'(x) = \frac{\pi}{3}\left[(r^2 - x^2)(1) + (r+x)(-2x)\right]$

$= \frac{\pi}{3}\left[(r+x)(r-x-2x)\right] = \frac{\pi}{3}(r+x)(r-3x)$

$= 0$ when $x = -r$ or $x = r/3$.

Now $V(r) = 0 = V(-r)$, so the maximum occurs at $x = r/3$ and the volume is $V\left(\frac{r}{3}\right) = \frac{\pi}{3}\left(r^2 - \frac{r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^3}{81}$.

55.



We minimize

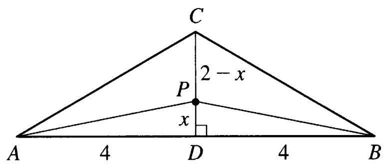
$L(x) = |PA| + |PB| + |PC| = 2\sqrt{x^2+16} + (5-x),$

$0 \le x \le 5$. $L'(x) = 2x/\sqrt{x^2+16} - 1 = 0 \quad \Leftrightarrow$

$2x = \sqrt{x^2+16} \quad \Leftrightarrow \quad 4x^2 = x^2 + 16 \quad \Leftrightarrow \quad x = \frac{4}{\sqrt{3}}$.

$L(0) = 13$, $L\left(\frac{4}{\sqrt{3}}\right) \approx 11.9$, $L(5) \approx 12.8$, so the minimum occurs when $x = \frac{4}{\sqrt{3}} \approx 2.3$.

56.

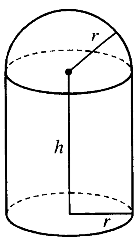


If $|CD| = 2$, the last part of $L(x)$ changes from $(5-x)$ to $(2-x)$ with $0 \le x \le 2$. But we still get $L'(x) = 0 \quad \Leftrightarrow$ $x = \frac{4}{\sqrt{3}}$, which isn't in the interval $[0, 2]$. Now $L(0) = 10$ and $L(2) = 2\sqrt{20} = 4\sqrt{5} \approx 8.9$. The minimum occurs when $P = C$.

57. $v = K\sqrt{\dfrac{L}{C} + \dfrac{C}{L}} \quad \Rightarrow \quad \dfrac{dv}{dL} = \dfrac{K}{2\sqrt{(L/C)+(C/L)}}\left(\dfrac{1}{C} - \dfrac{C}{L^2}\right) = 0 \quad \Leftrightarrow \quad \dfrac{1}{C} = \dfrac{C}{L^2} \quad \Leftrightarrow \quad L^2 = C^2 \quad \Leftrightarrow$

$L = C$. This gives the minimum velocity since $v' < 0$ for $0 < L < C$ and $v' > 0$ for $L > C$.

58.



We minimize the surface area $S = \pi r^2 + 2\pi rh + \frac{1}{2}(4\pi r^2) = 3\pi r^2 + 2\pi rh$.

Solving $V = \pi r^2 h + \frac{2}{3}\pi r^3$ for h, we get $h = \dfrac{V - \frac{2}{3}\pi r^3}{\pi r^2} = \dfrac{V}{\pi r^2} - \frac{2}{3}r$, so

$S(r) = 3\pi r^2 + 2\pi r\left[\dfrac{V}{\pi r^2} - \frac{2}{3}r\right] = \frac{5}{3}\pi r^2 + \dfrac{2V}{r}$.

$S'(r) = -\dfrac{2V}{r^2} + \frac{10}{3}\pi r = \dfrac{\frac{10}{3}\pi r^3 - 2V}{r^2} = 0 \quad \Leftrightarrow \quad \frac{10}{3}\pi r^3 = 2V$

$\Leftrightarrow \quad r^3 = \dfrac{3V}{5\pi} \quad \Leftrightarrow \quad r = \sqrt[3]{\dfrac{3V}{5\pi}}$. This gives an absolute minimum since $S'(r) < 0$ for $0 < r < \sqrt[3]{\dfrac{3V}{5\pi}}$ and

$S'(r) > 0$ for $r > \sqrt[3]{\dfrac{3V}{5\pi}}$. Thus, $h = \dfrac{V - \frac{2}{3}\pi \cdot \dfrac{3V}{5\pi}}{\pi\sqrt[3]{\dfrac{(3V)^2}{(5\pi)^2}}} = \dfrac{\left(V - \frac{2}{5}V\right)\sqrt[3]{(5\pi)^2}}{\pi\sqrt[3]{(3V)^2}} = \dfrac{3V\sqrt[3]{(5\pi)^2}}{5\pi\sqrt[3]{(3V)^2}} = \sqrt[3]{\dfrac{3V}{5\pi}} = r$.

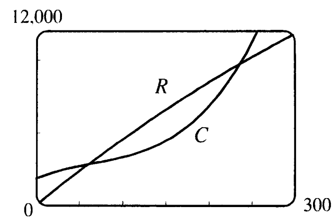
59. Let x denote the number of \$1 decreases in ticket price. Then the ticket price is $\$12 - \$1(x)$, and the average attendance is $11{,}000 + 1000(x)$. Now the revenue per game is

$$R(x) = (\text{price per person}) \times (\text{number of people per game})$$
$$= (12 - x)(11{,}000 + 1000x) = -1000x^2 + 1000x + 132{,}000$$

for $0 \le x \le 4$ (since the seating capacity is 15,000) $\Rightarrow$ $R'(x) = -2000x + 1000 = 0$ $\Leftrightarrow$ $x = 0.5$. This is a maximum since $R''(x) = -2000 < 0$ for all x. Now we must check the value of $R(x) = (12 - x)(11{,}000 + 1000x)$ at $x = 0.5$ and at the endpoints of the domain to see which value of x gives the maximum value of R. $R(0) = (12)(11{,}000) = 132{,}000$, $R(0.5) = (11.5)(11{,}500) = 132{,}250$, and $R(4) = (8)(15{,}000) = 120{,}000$. Thus, the maximum revenue of \$132,250 per game occurs when the average attendance is 11,500 and the ticket price is \$11.50.

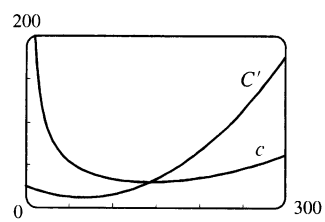
60. (a) $C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$ and $R(x) = xp(x) = 48.2x - 0.03x^2$. The profit is maximized when $C'(x) = R'(x)$.

From the figure, we estimate that the tangents are parallel when $x \approx 160$.



(b) $C'(x) = 25 - 0.4x + 0.003x^2$ and $R'(x) = 48.2 - 0.06x$. $C'(x) = R'(x)$ $\Rightarrow$ $0.003x^2 - 0.34x - 23.2 = 0$ $\Rightarrow$ $x_1 \approx 161.3$ $(x > 0)$. $R''(x) = -0.06$ and $C''(x) = -0.4 + 0.006x$, so $R''(x_1) = -0.06 < C''(x_1) \approx 0.57$ $\Rightarrow$ profit is maximized by producing 161 units.

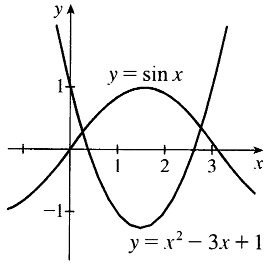
(c) $c(x) = \dfrac{C(x)}{x} = \dfrac{1800}{x} + 25 - 0.2x + 0.001x^2$ is the average cost. Since the average cost is minimized when the marginal cost equals the average cost, we graph $c(x)$ and $C'(x)$ and estimate the point of intersection. From the figure, $C'(x) = c(x)$ $\Leftrightarrow$ $x \approx 144$.



61. $f(x) = x^5 - x^4 + 3x^2 - 3x - 2$ $\Rightarrow$ $f'(x) = 5x^4 - 4x^3 + 6x - 3$, so

$x_{n+1} = x_n - \dfrac{x_n^5 - x_n^4 + 3x_n^2 - 3x_n - 2}{5x_n^4 - 4x_n^3 + 6x_n - 3}$. Now $x_1 = 1$ $\Rightarrow$ $x_2 = 1.5$ $\Rightarrow$ $x_3 \approx 1.343860$ $\Rightarrow$ $x_4 \approx 1.300320$ $\Rightarrow$ $x_5 \approx 1.297396$ $\Rightarrow$ $x_6 \approx 1.297383 \approx x_7$, so the root in $[1, 2]$ is 1.297383, to six decimal places.

62. Graphing $y = \sin x$ and $y = x^2 - 3x + 1$ shows that there are two roots, one about 0.3 and the other about 2.8.



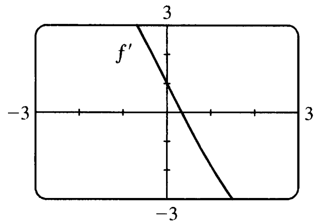
$f(x) = \sin x - x^2 + 3x - 1 \quad \Rightarrow \quad f'(x) = \cos x - 2x + 3 \quad \Rightarrow$

$x_{n+1} = x_n - \dfrac{\sin x_n - x_n^2 + 3x_n - 1}{\cos x_n - 2x_n + 3}$. Now $x_1 = 0.3 \quad \Rightarrow$

$x_2 \approx 0.268552 \quad \Rightarrow \quad x_3 \approx 0.268881 \approx x_4$ and $x_1 = 2.8 \quad \Rightarrow$

$x_2 \approx 2.770354 \quad \Rightarrow \quad x_3 \approx 2.770058 \approx x_4$, so to six decimal places, the roots are 0.268881 and 2.770058.

63. $f(t) = \cos t + t - t^2 \quad \Rightarrow \quad f'(t) = -\sin t + 1 - 2t$. $f'(t)$ exists for all t, so to find the maximum of f, we can examine the zeros of f'. From the graph of f', we see that a good choice for t_1 is $t_1 = 0.3$. Use $g(t) = -\sin t + 1 - 2t$ and $g'(t) = -\cos t - 2$ to obtain $t_2 \approx 0.33535293$, $t_3 \approx 0.33541803 \approx t_4$. Since $f''(t) = -\cos t - 2 < 0$ for all t, $f(0.33541803) \approx 1.16718557$ is the absolute maximum.

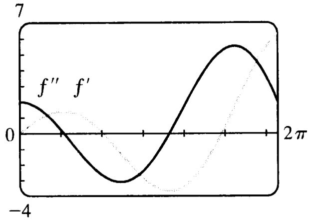


64. $y = f(x) = x \sin x$, $0 \le x \le 2\pi$. **A.** $D = [0, 2\pi]$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $\sin x = 0 \quad \Leftrightarrow \quad x = 0, \pi$, or 2π. **C.** There is no symmetry on D, but if f is defined for all real numbers x, then f is an even function. **D.** No asymptote **E.** $f'(x) = x\cos x + \sin x$. To find critical numbers in $(0, 2\pi)$, we graph f' and see that there are two critical numbers, about 2 and 4.9. To find them more precisely, we use Newton's method, setting $g(x) = f'(x) = x \cos x + \sin x$, so that $g'(x) = f''(x) = 2\cos x - x \sin x$ and

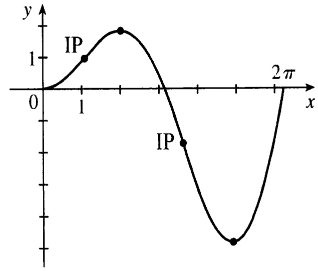
$x_{n+1} = x_n - \dfrac{x_n \cos x_n + \sin x_n}{2\cos x_n - x_n \sin x_n}$. $x_1 = 2 \quad \Rightarrow \quad x_2 \approx 2.029048$, $x_3 \approx 2.028758 \approx x_4$ and $x_1 = 4.9 \quad \Rightarrow$

$x_2 \approx 4.913214$, $x_3 \approx 4.913180 \approx x_4$, so the critical numbers, to six decimal places, are $r_1 = 2.028758$ and $r_2 = 4.913180$. By checking sample values of f' in $(0, r_1)$, (r_1, r_2), and $(r_2, 2\pi)$, we see that f is increasing on $(0, r_1)$, decreasing on (r_1, r_2), and increasing on $(r_2, 2\pi)$. **F.** Local maximum value $f(r_1) \approx 1.819706$, local minimum value $f(r_2) \approx -4.814470$. **G.** $f''(x) = 2\cos x - x \sin x$. To find points where $f''(x) = 0$, we graph f'' and find that $f''(x) = 0$ at about 1 and 3.6. To find the values more precisely, we use Newton's method. Set $h(x) = f''(x) = 2\cos x - x \sin x$. Then $h'(x) = -3\sin x - x \cos x$, so $x_{n+1} = x_n - \dfrac{2\cos x_n - x_n \sin x_n}{-3\sin x_n - x_n \cos x_n}$.

$x_1 = 1 \quad \Rightarrow \quad x_2 \approx 1.078028$, $x_3 \approx 1.076874 \approx x_4$ and $x_1 = 3.6 \quad \Rightarrow \quad x_2 \approx 3.643996$, $x_3 \approx 3.643597 \approx x_4$, so the zeros of f'', to six decimal places, are $r_3 = 1.076874$ and $r_4 = 3.643597$. By checking sample values of f'' in $(0, r_3)$, (r_3, r_4), and $(r_4, 2\pi)$, we see that f is CU on $(0, r_3)$, CD on (r_3, r_4), and CU on $(r_4, 2\pi)$. f has inflection points at $(r_3, f(r_3) \approx 0.948166)$ and $(r_4, f(r_4) \approx -1.753240)$.



H.



65. $f'(x) = \sqrt{x^5} - 4/\sqrt[5]{x} = x^{5/2} - 4x^{-1/5} \quad \Rightarrow \quad f(x) = \frac{2}{7}x^{7/2} - 4\left(\frac{5}{4}x^{4/5}\right) + C = \frac{2}{7}x^{7/2} - 5x^{4/5} + C$

66. $f'(x) = 8x - 3\sec^2 x \quad \Rightarrow \quad f(x) = 8\left(\frac{1}{2}x^2\right) - 3\tan x + C_n = 4x^2 - 3\tan x + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

67. $f'(x) = e^x - (2/\sqrt{x}) = e^x - 2x^{-1/2} \quad \Rightarrow$

$$f(x) = e^x - 2\frac{x^{-1/2+1}}{-1/2+1} + C = e^x - 2\frac{x^{1/2}}{1/2} + C = e^x - 4\sqrt{x} + C$$

68. $f'(x) = 2/(1+x^2) \quad \Rightarrow \quad f(x) = 2\arctan x + C$.

$f(0) = 2\arctan 0 + C = 0 + C = C$ and $f(0) = -1 \quad \Rightarrow \quad C = -1$, so $f(x) = 2\arctan x - 1$.

69. $f'(t) = 2t - 3\sin t \quad \Rightarrow \quad f(t) = t^2 + 3\cos t + C$.

$f(0) = 3 + C$ and $f(0) = 5 \quad \Rightarrow \quad C = 2$, so $f(t) = t^2 + 3\cos t + 2$.

70. $f'(u) = \dfrac{u^2 + \sqrt{u}}{u} = u + u^{-1/2} \quad \Rightarrow \quad f(u) = \frac{1}{2}u^2 + 2u^{1/2} + C$.

$f(1) = \frac{1}{2} + 2 + C$ and $f(1) = 3 \quad \Rightarrow \quad C = \frac{1}{2}$, so $f(u) = \frac{1}{2}u^2 + 2\sqrt{u} + \frac{1}{2}$.

71. $f''(x) = 1 - 6x + 48x^2 \quad \Rightarrow \quad f'(x) = x - 3x^2 + 16x^3 + C$. $f'(0) = C$ and $f'(0) = 2 \quad \Rightarrow \quad C = 2$, so $f'(x) = x - 3x^2 + 16x^3 + 2$ and hence, $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + D$. $f(0) = D$ and $f(0) = 1 \quad \Rightarrow$ $D = 1$, so $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + 1$.

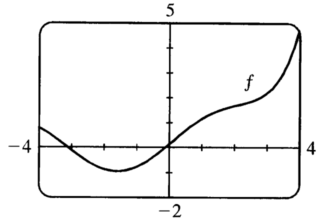
72. $f''(x) = 2x^3 + 3x^2 - 4x + 5 \quad \Rightarrow \quad f'(x) = \frac{1}{2}x^4 + x^3 - 2x^2 + 5x + C \quad \Rightarrow$

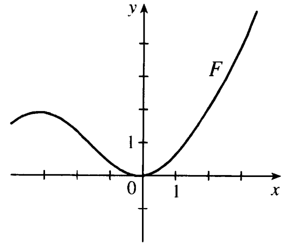
$f(x) = \frac{1}{10}x^5 + \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 + Cx + D$. $f(0) = D$ and $f(0) = 2 \quad \Rightarrow \quad D = 2$.

$f(1) = \frac{1}{10} + \frac{1}{4} - \frac{2}{3} + \frac{5}{2} + C + 2$ and $f(1) = 0 \quad \Rightarrow \quad C = -\frac{6}{60} - \frac{15}{60} + \frac{40}{60} - \frac{150}{60} - \frac{120}{60} = -\frac{251}{60}$, so

$f(x) = \frac{1}{10}x^5 + \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 - \frac{251}{60}x + 2$.

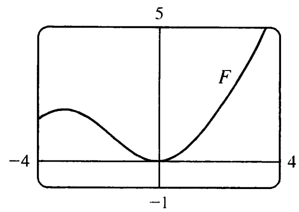
73. (a) Since f is 0 just to the left of the y-axis, we must have a minimum of F at the same place since we are increasing through $(0, 0)$ on F. There must be a local maximum to the left of $x = -3$, since f changes from positive to negative there.





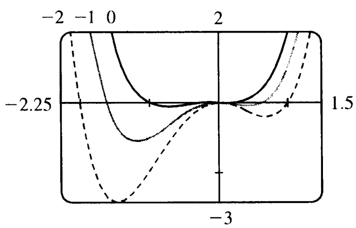
(b) $f(x) = 0.1e^x + \sin x \quad \Rightarrow \quad F(x) = 0.1e^x - \cos x + C$.
$F(0) = 0 \quad \Rightarrow \quad 0.1 - 1 + C = 0 \quad \Rightarrow \quad C = 0.9$, so
$F(x) = 0.1e^x - \cos x + 0.9$.

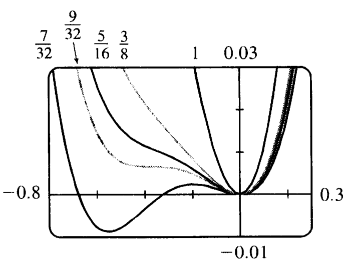
(c)



74. $f(x) = x^4 + x^3 + cx^2 \quad \Rightarrow \quad f'(x) = 4x^3 + 3x^2 + 2cx$. This is 0 when $x(4x^2 + 3x + 2c) = 0 \quad \Leftrightarrow$ $x = 0$ or $4x^2 + 3x + 2c = 0$. Using the quadratic formula, we find that the roots of this last equation are $x = \dfrac{-3 \pm \sqrt{9 - 32c}}{8}$. Now if $9 - 32c < 0 \quad \Leftrightarrow \quad c > \frac{9}{32}$, then $(0, 0)$ is the only critical point, a minimum. If $c = \frac{9}{32}$, then there are two critical points (a minimum at $x = 0$, and a horizontal tangent with no maximum or minimum at $x = -\frac{3}{8}$) and if $c < \frac{9}{32}$, then there are three critical points except when $c = 0$, in which case the root with the $+$ sign coincides with the critical point at $x = 0$. For $0 < c < \frac{9}{32}$, there is a minimum at $x = -\dfrac{3}{8} - \dfrac{\sqrt{9 - 32c}}{8}$, a maximum at $x = -\dfrac{3}{8} + \dfrac{\sqrt{9 - 32c}}{8}$, and a minimum at $x = 0$. For $c = 0$, there is a minimum at $x = -\frac{3}{4}$ and a horizontal tangent with no extremum at $x = 0$, and for $c < 0$, there is a maximum at $x = 0$, and there are minima at $x = -\dfrac{3}{8} \pm \dfrac{\sqrt{9 - 32c}}{8}$. Now we calculate $f''(x) = 12x^2 + 6x + 2c$. The roots of this equation are $x = \dfrac{-6 \pm \sqrt{36 - 4 \cdot 12 \cdot 2c}}{24}$. So if $36 - 96c \leq 0 \quad \Leftrightarrow \quad c \geq \frac{3}{8}$, then there is no inflection point. If $c < \frac{3}{8}$, then there are two inflection points at $x = -\dfrac{1}{4} \pm \dfrac{\sqrt{9 - 24c}}{12}$.

Value of c	No. of CP	No. of IP
$c < 0$	3	2
$c = 0$	2	2
$0 < c < \frac{9}{32}$	3	2
$c = \frac{9}{32}$	2	2
$\frac{9}{32} < c < \frac{3}{8}$	1	2
$c \geq \frac{3}{8}$	1	0

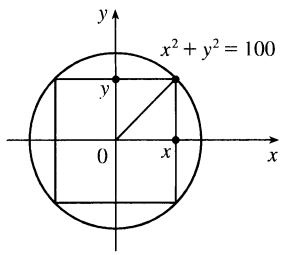




75. Choosing the positive direction to be upward, we have $a(t) = -9.8 \;\Rightarrow\; v(t) = -9.8t + v_0$, but $v(0) = 0 = v_0$ $\Rightarrow\; v(t) = -9.8t = s'(t) \;\Rightarrow\; s(t) = -4.9t^2 + s_0$, but $s(0) = s_0 = 500 \;\Rightarrow\; s(t) = -4.9t^2 + 500$. When $s = 0$, $-4.9t^2 + 500 = 0 \;\Rightarrow\; t_1 = \sqrt{\frac{500}{4.9}} \approx 10.1 \;\Rightarrow\; v(t_1) = -9.8\sqrt{\frac{500}{4.9}} \approx -98.995$ m/s. Since the canister has been designed to withstand an impact velocity of 100 m/s, the canister will *not burst*.

76. Let $s_A(t)$ and $s_B(t)$ be the position functions for cars A and B and let $f(t) = s_A(t) - s(t)$. Since A passed B twice, there must be three values of t such that $f(t) = 0$. Then by three applications of Rolle's Theorem (see Exercise 4.2.22), there is a number c such that $f''(c) = 0$. So $s''_A(c) = s''_B(c)$, that is, A and B had equal accelerations at $t = c$. We assume that f is continuous on $[0, T]$ and twice differentiable on $(0, T)$, where T is the total time of the race.

77. (a)



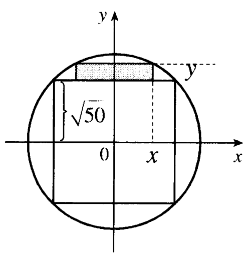
The cross-sectional area of the rectangular beam is

$A = 2x \cdot 2y = 4xy = 4x\sqrt{100 - x^2}$, $0 \le x \le 10$, so

$$\frac{dA}{dx} = 4x\left(\tfrac{1}{2}\right)\left(100 - x^2\right)^{-1/2}(-2x) + \left(100 - x^2\right)^{1/2} \cdot 4 = \frac{-4x^2}{(100 - x^2)^{1/2}} + 4\left(100 - x^2\right)^{1/2} = \frac{4\left[-x^2 + (100 - x^2)\right]}{(100 - x^2)^{1/2}}.$$

$\frac{dA}{dx} = 0$ when $-x^2 + \left(100 - x^2\right) = 0 \;\Rightarrow\; x^2 = 50 \;\Rightarrow\; x = \sqrt{50} \approx 7.07 \;\Rightarrow$ $y = \sqrt{100 - \left(\sqrt{50}\right)^2} = \sqrt{50}$. Since $A(0) = A(10) = 0$, the rectangle of maximum area is a square.

(b)



The cross-sectional area of each rectangular plank (shaded in the figure) is

$A = 2x\left(y - \sqrt{50}\right) = 2x\left[\sqrt{100 - x^2} - \sqrt{50}\right]$, $0 \le x \le \sqrt{50}$, so

$$\frac{dA}{dx} = 2\left(\sqrt{100 - x^2} - \sqrt{50}\right) + 2x\left(\tfrac{1}{2}\right)\left(100 - x^2\right)^{-1/2}(-2x) = 2\left(100 - x^2\right)^{1/2} - 2\sqrt{50} - \frac{2x^2}{(100 - x^2)^{1/2}}$$

Set $\frac{dA}{dx} = 0$: $\left(100 - x^2\right) - \sqrt{50}\left(100 - x^2\right)^{1/2} - x^2 = 0 \;\Rightarrow\; 100 - 2x^2 = \sqrt{50}\left(100 - x^2\right)^{1/2} \;\Rightarrow$ $10{,}000 - 400x^2 + 4x^4 = 50\left(100 - x^2\right) \;\Rightarrow\; 4x^4 - 350x^2 + 5000 = 0 \;\Rightarrow$ $2x^4 - 175x^2 + 2500 = 0 \;\Rightarrow\; x^2 = \frac{175 \pm \sqrt{10{,}625}}{4} \approx 69.52$ or $17.98 \;\Rightarrow\; x \approx 8.34$ or 4.24.

But $8.34 > \sqrt{50}$, so $x_1 \approx 4.24 \;\Rightarrow\; y - \sqrt{50} = \sqrt{100 - x_1^2} - \sqrt{50} \approx 1.99$. Each plank should have dimensions about $8\frac{1}{2}$ inches by 2 inches.

(c) From the figure in part (a), the width is $2x$ and the depth is $2y$, so the strength is

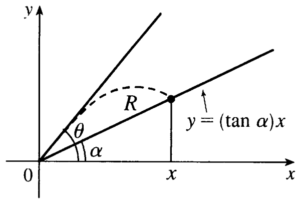
$S = k(2x)(2y)^2 = 8kxy^2 = 8kx(100 - x^2) = 800kx - 8kx^3$, $0 \le x \le 10$. $dS/dx = 800k - 24kx^2 = 0$

when $24kx^2 = 800k \Rightarrow x^2 = \frac{100}{3} \Rightarrow x = \frac{10}{\sqrt{3}} \Rightarrow y = \sqrt{\frac{200}{3}} = \frac{10\sqrt{2}}{\sqrt{3}} = \sqrt{2}\,x$. Since

$S(0) = S(10) = 0$, the maximum strength occurs when $x = \frac{10}{\sqrt{3}}$. The dimensions should be

$\frac{20}{\sqrt{3}} \approx 11.55$ inches by $\frac{20\sqrt{2}}{\sqrt{3}} \approx 16.33$ inches.

78. (a)



$y = (\tan\theta)x - \dfrac{g}{2v^2\cos^2\theta}x^2$. The parabola intersects the line when $(\tan\alpha)x = (\tan\theta)x - \dfrac{g}{2v^2\cos^2\theta}x^2 \Rightarrow$

$$x = \frac{(\tan\theta - \tan\alpha)2v^2\cos^2\theta}{g} \Rightarrow$$

$$R(\theta) = \frac{x}{\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)\frac{2v^2\cos^2\theta}{g\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)(\cos\theta\,\cos\alpha)\frac{2v^2\cos\theta}{g\cos^2\alpha}$$

$$= (\sin\theta\,\cos\alpha - \sin\alpha\,\cos\theta)\frac{2v^2\cos\theta}{g\cos^2\alpha} = \sin(\theta - \alpha)\frac{2v^2\cos\theta}{g\cos^2\alpha}$$

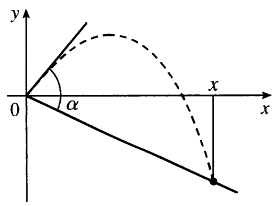
(b) $$R'(\theta) = \frac{2v^2}{g\cos^2\alpha}[\cos\theta\cdot\cos(\theta - \alpha) + \sin(\theta - \alpha)(-\sin\theta)] = \frac{2v^2}{g\cos^2\alpha}\cos[\theta + (\theta - \alpha)]$$

$$= \frac{2v^2}{g\cos^2\alpha}\cos(2\theta - \alpha) = 0 \text{ when } \cos(2\theta - \alpha) = 0 \Rightarrow 2\theta - \alpha = \tfrac{\pi}{2} \Rightarrow$$

$\theta = \dfrac{\pi/2 + \alpha}{2} = \dfrac{\pi}{4} + \dfrac{\alpha}{2}$. The First Derivative Test shows that this gives a maximum value for $R(\theta)$.

[This could be done without calculus by applying the formula for $\sin x \cos y$ to $R(\theta)$.]

(c)



Replacing α by $-\alpha$ in part (a), we get $R(\theta) = \dfrac{2v^2\cos\theta\,\sin(\theta + \alpha)}{g\cos^2\alpha}$.

Proceeding as in part (b), or simply by replacing α by $-\alpha$ in the result of part (b), we see that $R(\theta)$ is maximized when $\theta = \dfrac{\pi}{4} - \dfrac{\alpha}{2}$.

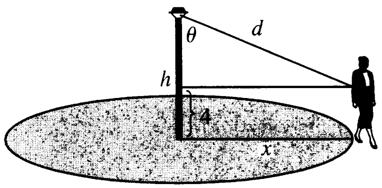
79. (a) $$I = \frac{k\cos\theta}{d^2} = \frac{k(h/d)}{d^2} = k\frac{h}{d^3} = k\frac{h}{\left(\sqrt{40^2 + h^2}\right)^3} = k\frac{h}{(1600 + h^2)^{3/2}} \Rightarrow$$

$$\frac{dI}{dh} = k\frac{(1600 + h^2)^{3/2} - h\frac{3}{2}(1600 + h^2)^{1/2}\cdot 2h}{\left[(1600 + h^2)^{3/2}\right]^2} = \frac{k(1600 + h^2)^{1/2}(1600 + h^2 - 3h^2)}{(1600 + h^2)^3}$$

$$= \frac{k(1600 - 2h^2)}{(1600 + h^2)^{5/2}} \quad [k \text{ is the constant of proportionality}]$$

Set $dI/dh = 0$: $1600 - 2h^2 = 0 \Rightarrow h^2 = 800 \Rightarrow h = \sqrt{800} = 20\sqrt{2}$. By the First Derivative Test, I has a local maximum at $h = 20\sqrt{2} \approx 28$ ft.

(b)



$$\frac{dx}{dt} = 4 \text{ ft/s}$$

$$I = \frac{k\cos\theta}{d^2} = \frac{k[(h-4)/d]}{d^2} = \frac{k(h-4)}{d^3} = \frac{k(h-4)}{[(h-4)^2+x^2]^{3/2}} = k(h-4)\left[(h-4)^2+x^2\right]^{-3/2}$$

$$\frac{dI}{dt} = \frac{dI}{dx}\cdot\frac{dx}{dt} = k(h-4)\left(-\tfrac{3}{2}\right)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 2x\cdot\frac{dx}{dt}$$

$$= k(h-4)(-3x)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 4 = \frac{-12xk(h-4)}{[(h-4)^2+x^2]^{5/2}}$$

$$\left.\frac{dI}{dt}\right|_{x=40} = -\frac{480k(h-4)}{[(h-4)^2+1600]^{5/2}}$$

80. (a) $V'(t)$ is the rate of change of the volume of the water with respect to time. $H'(t)$ is the rate of change of the height of the water with respect to time. Since the volume and the height are increasing, $V'(t)$ and $H'(t)$ are positive.

(b) $V'(t)$ is constant, so $V''(t)$ is zero (the slope of a constant function is 0).

(c) At first, the height H of the water increases quickly because the tank is narrow. But as the sphere widens, the rate of increase of the height slows down, reaching a minimum at $t = t_2$. Thus, the height is increasing at a decreasing rate on $(0, t_2)$, so its graph is concave downward and $H''(t_1) < 0$. As the sphere narrows for $t > t_2$, the rate of increase of the height begins to increase, and the graph of H is concave upward. Therefore, $H''(t_2) = 0$ and $H''(t_3) > 0$.

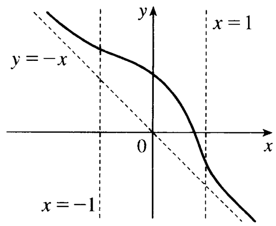
81. We first show that $\dfrac{x}{1+x^2} < \tan^{-1} x$ for $x > 0$. Let $f(x) = \tan^{-1} x - \dfrac{x}{1+x^2}$. Then

$f'(x) = \dfrac{1}{1+x^2} - \dfrac{1(1+x^2) - x(2x)}{(1+x^2)^2} = \dfrac{(1+x^2)-(1-x^2)}{(1+x^2)^2} = \dfrac{2x^2}{(1+x^2)^2} > 0$ for $x > 0$. So $f(x)$ is increasing on $(0,\infty)$. Hence, $0 < x \;\Rightarrow\; 0 = f(0) < f(x) = \tan^{-1} x - \dfrac{x}{1+x^2}$. So $\dfrac{x}{1+x^2} < \tan^{-1} x$ for $0 < x$. We next show that $\tan^{-1} x < x$ for $x > 0$. Let $h(x) = x - \tan^{-1} x$. Then

$h'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{x^2}{1+x^2} > 0$. Hence, $h(x)$ is increasing on $(0,\infty)$. So for $0 < x$,

$0 = h(0) < h(x) = x - \tan^{-1} x$. Hence, $\tan^{-1} x < x$ for $x > 0$, and we conclude that $\dfrac{x}{1+x^2} < \tan^{-1} x < x$ for $x > 0$.

82. If $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim_{x \to \pm\infty} [f(x) + x] = 0$, then f is decreasing everywhere, concave up on $(-\infty, -1)$ and $(1, \infty)$, concave down on $(-1, 1)$, and approaches the line $y = -x$ as $x \to \pm\infty$. An example of such a graph is sketched.



PROBLEMS PLUS

1. Let $y = f(x) = e^{-x^2}$. The area of the rectangle under the curve from $-x$ to x is $A(x) = 2xe^{-x^2}$ where $x \geq 0$. We maximize $A(x)$: $A'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2e^{-x^2}(1 - 2x^2) = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$. This gives a maximum since $A'(x) > 0$ for $0 \leq x < \frac{1}{\sqrt{2}}$ and $A'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$. We next determine the points of inflection of $f(x)$. Notice that $f'(x) = -2xe^{-x^2} = -A(x)$. So $f''(x) = -A'(x)$ and hence, $f''(x) < 0$ for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and $f''(x) > 0$ for $x < -\frac{1}{\sqrt{2}}$ and $x > \frac{1}{\sqrt{2}}$. So $f(x)$ changes concavity at $x = \pm\frac{1}{\sqrt{2}}$, and the two vertices of the rectangle of largest area are at the inflection points.

2. Let $f(x) = \sin x - \cos x$ on $[0, 2\pi]$ since f has period 2π. $f'(x) = \cos x + \sin x = 0 \Leftrightarrow \cos x = -\sin x \Leftrightarrow \tan x = -1 \Leftrightarrow x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. Evaluating f at its critical numbers and endpoints, we get $f(0) = -1$, $f\left(\frac{3\pi}{4}\right) = \sqrt{2}$, $f\left(\frac{7\pi}{4}\right) = -\sqrt{2}$, and $f(2\pi) = -1$. So f has absolute maximum value $\sqrt{2}$ and absolute minimum value $-\sqrt{2}$. Thus, $-\sqrt{2} \leq \sin x - \cos x \leq \sqrt{2} \Rightarrow |\sin x - \cos x| \leq \sqrt{2}$.

3. First, we recognize some symmetry in the inequality: $\dfrac{e^{x+y}}{xy} \geq e^2 \Leftrightarrow \dfrac{e^x}{x} \cdot \dfrac{e^y}{y} \geq e \cdot e$. This suggests that we need to show that $\dfrac{e^x}{x} \geq e$ for $x > 0$. If we can do this, then the inequality $\dfrac{e^y}{y} \geq e$ is true, and the given inequality follows. $f(x) = \dfrac{e^x}{x} \Rightarrow f'(x) = \dfrac{xe^x - e^x}{x^2} = \dfrac{e^x(x-1)}{x^2} = 0 \Rightarrow x = 1$. By the First Derivative Test, we have a minimum of $f(1) = e$, so $e^x/x \geq e$ for all x.

4. $x^2y^2(4 - x^2)(4 - y^2) = x^2(4 - x^2)y^2(4 - y^2) = f(x)f(y)$, where $f(t) = t^2(4 - t^2)$. We will show that $0 \leq f(t) \leq 4$ for $|t| \leq 2$, which gives $0 \leq f(x)f(y) \leq 16$ for $|x| \leq 2$ and $|y| \leq 2$.

$f(t) = 4t^2 - t^4 \Rightarrow f'(t) = 8t - 4t^3 = 4t(2 - t^2) = 0 \Rightarrow t = 0$ or $\pm\sqrt{2}$.

$f(0) = 0$, $f(\pm\sqrt{2}) = 2(4 - 2) = 4$, and $f(2) = 0$. So 0 is the absolute minimum value of $f(t)$ on $[-2, 2]$ and 4 is the absolute maximum value of $f(t)$ on $[-2, 2]$. We conclude that $0 \leq f(t) \leq 4$ for $|t| \leq 2$ and hence,

$0 \leq f(x)f(y) \leq 4^2$ or $0 \leq x^2(4 - x^2)y^2(4 - y^2) \leq 16$.

5. First we show that $x(1 - x) \leq \frac{1}{4}$ for all x. Let $f(x) = x(1 - x) = x - x^2$. Then $f'(x) = 1 - 2x$. This is 0 when $x = \frac{1}{2}$ and $f'(x) > 0$ for $x < \frac{1}{2}$, $f'(x) < 0$ for $x > \frac{1}{2}$, so the absolute maximum of f is $f\left(\frac{1}{2}\right) = \frac{1}{4}$. Thus, $x(1 - x) \leq \frac{1}{4}$ for all x.

Now suppose that the given assertion is false, that is, $a(1 - b) > \frac{1}{4}$ and $b(1 - a) > \frac{1}{4}$. Multiply these inequalities: $a(1 - b)b(1 - a) > \frac{1}{16} \Rightarrow [a(1 - a)]\,[b(1 - b)] > \frac{1}{16}$. But we know that $a(1 - a) \leq \frac{1}{4}$ and $b(1 - b) \leq \frac{1}{4} \Rightarrow [a(1 - a)]\,[b(1 - b)] \leq \frac{1}{16}$. Thus, we have a contradiction, so the given assertion is proved.

6. Let $P(a, 1-a^2)$ be the point of contact. The equation of the tangent line at P is $y - (1-a^2) = (-2a)(x-a)$ $\Rightarrow$ $y - 1 + a^2 = -2ax + 2a^2$ $\Rightarrow$ $y = -2ax + a^2 + 1$. To find the x-intercept, put $y = 0$: $2ax = a^2 + 1$ $\Rightarrow$ $x = \dfrac{a^2+1}{2a}$. To find the y-intercept, put $x = 0$: $y = a^2 + 1$. Therefore, the area of the triangle is $\dfrac{1}{2}\left(\dfrac{a^2+1}{2a}\right)(a^2+1) = \dfrac{(a^2+1)^2}{4a}$. Therefore, we minimize the function $A(a) = \dfrac{(a^2+1)^2}{4a}$, $0 < a \le 1$.

$$A'(a) = \frac{(4a)2(a^2+1)(2a) - (a^2+1)^2(4)}{16a^2} = \frac{(a^2+1)[4a^2 - (a^2+1)]}{4a^2} = \frac{(a^2+1)(3a^2-1)}{4a^2}.$$

$A'(a) = 0$ when $3a^2 - 1 = 0$ $\Rightarrow$ $a = \frac{1}{\sqrt{3}}$. $A'(a) < 0$ for $a < \frac{1}{\sqrt{3}}$, $A'(a) > 0$ for $a > \frac{1}{\sqrt{3}}$. So by the First Derivative Test, there is an absolute minimum when $a = \frac{1}{\sqrt{3}}$. The required point is $\left(\frac{1}{\sqrt{3}}, \frac{2}{3}\right)$ and the corresponding minimum area is $A\left(\frac{1}{\sqrt{3}}\right) = \frac{4\sqrt{3}}{9}$.

7. Differentiating $x^2 + xy + y^2 = 12$ implicitly with respect to x gives $2x + y + x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$. At a highest or lowest point, $\dfrac{dy}{dx} = 0$ $\Leftrightarrow$ $y = -2x$. Substituting $-2x$ for y in the original equation gives $x^2 + x(-2x) + (-2x)^2 = 12$, so $3x^2 = 12$ and $x = \pm 2$. If $x = 2$, then $y = -2x = -4$, and if $x = -2$ then $y = 4$. Thus, the highest and lowest points are $(-2, 4)$ and $(2, -4)$.

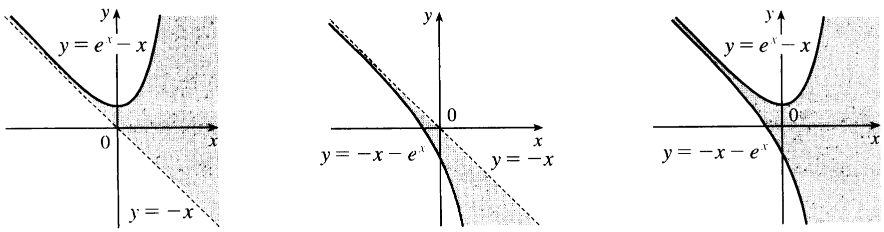
8. *Case (i) (first graph):* For $x + y \ge 0$, that is, $y \ge -x$, $|x+y| = x + y \le e^x$ $\Rightarrow$ $y \le e^x - x$.

Note that $y = e^x - x$ is always above the line $y = -x$ and that $y = -x$ is a slant asymptote.

Case (ii) (second graph): For $x + y < 0$, that is, $y < -x$, $|x+y| = -x - y \le e^x$ $\Rightarrow$ $y \ge -x - e^x$.

Note that $-x - e^x$ is always below the line $y = -x$ and $y = -x$ is a slant asymptote.

Putting the two pieces together gives the third graph.



9. $A = (x_1, x_1^2)$ and $B = (x_2, x_2^2)$, where x_1 and x_2 are the solutions of the quadratic equation $x^2 = mx + b$. Let $P = (x, x^2)$ and set $A_1 = (x_1, 0)$, $B_1 = (x_2, 0)$, and $P_1 = (x, 0)$. Let $f(x)$ denote the area of triangle PAB. Then $f(x)$ can be expressed in terms of the areas of three trapezoids as follows:

$$\begin{aligned} f(x) &= \text{area}\,(A_1ABB_1) - \text{area}\,(A_1APP_1) - \text{area}\,(B_1BPP_1) \\ &= \tfrac{1}{2}(x_1^2 + x_2^2)(x_2 - x_1) - \tfrac{1}{2}(x_1^2 + x^2)(x - x_1) - \tfrac{1}{2}(x^2 + x_2^2)(x_2 - x) \end{aligned}$$

After expanding and canceling terms, we get

$f(x) = \frac{1}{2}\left(x_2x_1^2 - x_1x_2^2 - xx_1^2 + x_1x^2 - x_2x^2 + xx_2^2\right) = \frac{1}{2}\left[x_1^2(x_2 - x) + x_2^2(x - x_1) + x^2(x_1 - x_2)\right]$

$f'(x) = \frac{1}{2}\left[-x_1^2 + x_2^2 + 2x(x_1 - x_2)\right]$. $f''(x) = \frac{1}{2}[2(x_1 - x_2)] = x_1 - x_2 < 0$ since $x_2 > x_1$.

$f'(x) = 0 \quad\Rightarrow\quad 2x(x_1 - x_2) = x_1^2 - x_2^2 \quad\Rightarrow\quad x_P = \frac{1}{2}(x_1 + x_2)$.

$$\begin{aligned}
f(x_P) &= \tfrac{1}{2}\left(x_1^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + x_2^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + \tfrac{1}{4}(x_1 + x_2)^2(x_1 - x_2)\right) \\
&= \tfrac{1}{2}\left[\tfrac{1}{2}(x_2 - x_1)\left(x_1^2 + x_2^2\right) - \tfrac{1}{4}(x_2 - x_1)(x_1 + x_2)^2\right] \\
&= \tfrac{1}{8}(x_2 - x_1)\left[2\left(x_1^2 + x_2^2\right) - \left(x_1^2 + 2x_1x_2 + x_2^2\right)\right] \\
&= \tfrac{1}{8}(x_2 - x_1)\left(x_1^2 - 2x_1x_2 + x_2^2\right) = \tfrac{1}{8}(x_2 - x_1)(x_1 - x_2)^2 = \tfrac{1}{8}(x_2 - x_1)(x_2 - x_1)^2 \\
&= \tfrac{1}{8}(x_2 - x_1)^3
\end{aligned}$$

To put this in terms of m and b, we solve the system $y = x_1^2$ and $y = mx_1 + b$, giving us $x_1^2 - mx_1 - b = 0 \quad\Rightarrow$ $x_1 = \frac{1}{2}\left(m - \sqrt{m^2 + 4b}\right)$. Similarly, $x_2 = \frac{1}{2}\left(m + \sqrt{m^2 + 4b}\right)$. The area is then $\frac{1}{8}(x_2 - x_1)^3 = \frac{1}{8}\left(\sqrt{m^2 + 4b}\right)^3$, and is attained at the point $P\left(x_P, x_P^2\right) = P\left(\frac{1}{2}m, \frac{1}{4}m^2\right)$.

Note: Another way to get an expression for $f(x)$ is to use the formula for an area of a triangle in terms of the coordinates of the vertices: $f(x) = \frac{1}{2}\left[\left(x_2x_1^2 - x_1x_2^2\right) + \left(x_1x^2 - xx_1^2\right) + \left(xx_2^2 - x_2x^2\right)\right]$.

10. If $f''(x) > 0$ for all x, then f' is increasing on $(-\infty, \infty)$, so $f'(0)$ must be greater than $f'(-1)$. But $f'(0) = 0 < \frac{1}{2} = f'(-1)$, so such a function cannot exist.

11. $f(x) = \left(a^2 + a - 6\right)\cos 2x + (a - 2)x + \cos 1 \quad\Rightarrow\quad f'(x) = -\left(a^2 + a - 6\right)\sin 2x\,(2) + (a - 2)$.

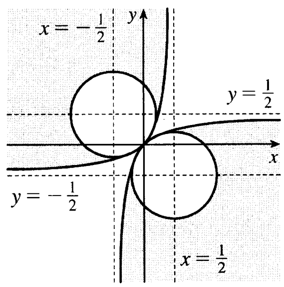
The derivative exists for all x, so the only possible critical points will occur where $f'(x) = 0 \quad\Leftrightarrow$ $2(a - 2)(a + 3)\sin 2x = a - 2 \quad\Leftrightarrow\quad$ either $a = 2$ or $2(a + 3)\sin 2x = 1$, with the latter implying that $\sin 2x = \dfrac{1}{2(a + 3)}$. Since the range of $\sin 2x$ is $[-1, 1]$, this equation has no solution whenever either $\dfrac{1}{2(a + 3)} < -1$ or $\dfrac{1}{2(a + 3)} > 1$. Solving these inequalities, we get $-\frac{7}{2} < a < -\frac{5}{2}$.

12. To sketch the region $\{(x, y) \mid 2xy \le |x - y| \le x^2 + y^2\}$, we consider two cases.

Case 1: $x \ge y$ This is the case in which (x, y) lies on or below the line $y = x$. The double inequality becomes $2xy \le x - y \le x^2 + y^2$. The right-hand inequality holds if and only if $x^2 - x + y^2 + y \ge 0 \quad\Leftrightarrow$ $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 \ge \frac{1}{2} \quad\Leftrightarrow\quad (x, y)$ lies on or outside the circle with radius $\frac{1}{\sqrt{2}}$ centered at $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

The left-hand inequality holds if and only if $2xy - x + y \le 0 \iff xy - \frac{1}{2}x + \frac{1}{2}y \le 0 \iff$ $\left(x + \frac{1}{2}\right)\left(y - \frac{1}{2}\right) \le -\frac{1}{4} \iff (x, y)$ lies on or below the hyperbola $\left(x + \frac{1}{2}\right)\left(y - \frac{1}{2}\right) = -\frac{1}{4}$, which passes through the origin and approaches the lines $y = \frac{1}{2}$ and $x = -\frac{1}{2}$ asymptotically.

Case 2: $y \ge x$ This is the case in which (x, y) lies on or above the line $y = x$. The double inequality becomes $2xy \le y - x \le x^2 + y^2$. The right-hand inequality holds if and only if $x^2 + x + y^2 - y \ge 0 \iff$ $\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \ge \frac{1}{2} \iff (x, y)$ lies on or outside the circle of radius $\frac{1}{\sqrt{2}}$ centered at $\left(-\frac{1}{2}, \frac{1}{2}\right)$. The left-hand inequality holds if and only if $2xy + x - y \le 0 \iff xy + \frac{1}{2}x - \frac{1}{2}y \le 0 \iff \left(x - \frac{1}{2}\right)\left(y + \frac{1}{2}\right) \le -\frac{1}{4}$ $\iff (x, y)$ lies on or above the left-hand branch of the hyperbola $\left(x - \frac{1}{2}\right)\left(y + \frac{1}{2}\right) = -\frac{1}{4}$, which passes through the origin and approaches the lines $y = -\frac{1}{2}$ and $x = \frac{1}{2}$ asymptotically. Therefore, the region of interest consists of the points on or above the left branch of the hyperbola $\left(x - \frac{1}{2}\right)\left(y + \frac{1}{2}\right) = -\frac{1}{4}$ that are on or outside the circle $\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$, together with the points on or below the right branch of the hyperbola $\left(x + \frac{1}{2}\right)\left(y - \frac{1}{2}\right) = -\frac{1}{4}$ that are on or outside the circle $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{2}$. Note that the inequalities are unchanged when x and y are interchanged, so the region is symmetric about the line $y = x$. So we need only have analyzed case 1 and then reflected that region about the line $y = x$, instead of considering case 2.

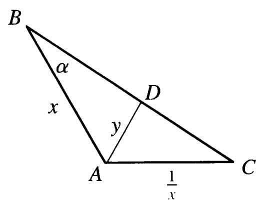


13. (a) Let $y = |AD|$, $x = |AB|$, and $1/x = |AC|$, so that $|AB| \cdot |AC| = 1$.

We compute the area $\mathcal{A}$ of $\triangle ABC$ in two ways. First,

$\mathcal{A} = \frac{1}{2}|AB|\,|AC| \sin\frac{2\pi}{3} = \frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$. Second,

$$\begin{aligned} \mathcal{A} &= (\text{area of } \triangle ABD) + (\text{area of } \triangle ACD) \\ &= \tfrac{1}{2}|AB|\,|AD| \sin\tfrac{\pi}{3} + \tfrac{1}{2}|AD|\,|AC| \sin\tfrac{\pi}{3} \\ &= \tfrac{1}{2}xy\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}y(1/x)\,\tfrac{\sqrt{3}}{2} = \tfrac{\sqrt{3}}{4}y(x + 1/x) \end{aligned}$$



Equating the two expressions for the area, we get $\frac{\sqrt{3}}{4}y\left(x + \frac{1}{x}\right) = \frac{\sqrt{3}}{4} \iff y = \frac{1}{x + 1/x} = \frac{x}{x^2 + 1}$, $x > 0$.

Another method: Use the Law of Sines on the triangles ABD and ABC. In $\triangle ABD$, we have $\angle A + \angle B + \angle D = 180^\circ \iff 60^\circ + \alpha + \angle D = 180^\circ \iff \angle D = 120^\circ - \alpha$. Thus,

$$\frac{x}{y} = \frac{\sin(120^\circ - \alpha)}{\sin\alpha} = \frac{\sin 120^\circ \cos\alpha - \cos 120^\circ \sin\alpha}{\sin\alpha} = \frac{\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha}{\sin\alpha} \Rightarrow \frac{x}{y} = \frac{\sqrt{3}}{2}\cot\alpha + \frac{1}{2},$$

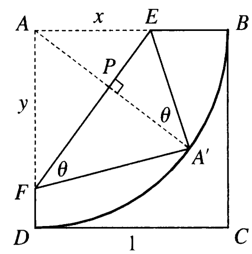
and by a similar argument with $\triangle ABC$, $\frac{\sqrt{3}}{2}\cot\alpha = x^2 + \frac{1}{2}$. Eliminating $\cot\alpha$ gives $\frac{x}{y} = \left(x^2 + \frac{1}{2}\right) + \frac{1}{2} \Rightarrow$

$$y = \frac{x}{x^2 + 1}, \; x > 0.$$

(b) We differentiate our expression for y with respect to x to find the maximum:

$\dfrac{dy}{dx} = \dfrac{(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} = 0$ when $x = 1$. This indicates a maximum by the First Derivative Test, since $y'(x) > 0$ for $0 < x < 1$ and $y'(x) < 0$ for $x > 1$, so the maximum value of y is $y(1) = \frac{1}{2}$.

14. Let $x = |AE|$, $y = |AF|$ as shown. The area $\mathcal{A}$ of the ΔAEF is $\mathcal{A} = \frac{1}{2}xy$. We need to find a relationship between x and y, so that we can take the derivative $d\mathcal{A}/dx$ and then find the maximum and minimum areas. Now let A' be the point on which A ends up after the fold has been performed, and let P be the intersection of AA' and EF. Note that AA' is perpendicular to EF since we are reflecting A through the line EF to get to A', and that $|AP| = |PA'|$ for the same reason. But $|AA'| = 1$, since AA' is a radius of the circle. Since $|AP| + |PA'| = |AA'|$,



we have $|AP| = \frac{1}{2}$. Another way to express the area of the triangle is

$\mathcal{A} = \frac{1}{2}|EF|\,|AP| = \frac{1}{2}\sqrt{x^2+y^2}\left(\frac{1}{2}\right) = \frac{1}{4}\sqrt{x^2+y^2}$. Equating the two expressions for $\mathcal{A}$, we get

$\frac{1}{2}xy = \frac{1}{4}\sqrt{x^2+y^2} \quad\Rightarrow\quad 4x^2y^2 = x^2+y^2 \quad\Rightarrow\quad y^2(4x^2-1) = x^2 \quad\Rightarrow\quad y = x/\sqrt{4x^2-1}$.

(Note that we could also have derived this result from the similarity of $\triangle A'PE$ and $\triangle A'FE$; that is,

$$\frac{|A'P|}{|PE|} = \frac{|A'F|}{|A'E|} \quad\Rightarrow\quad \frac{\frac{1}{2}}{\sqrt{x^2 - \left(\frac{1}{2}\right)^2}} = \frac{y}{x} \quad\Rightarrow\quad y = \frac{\frac{1}{2}x}{\sqrt{4x^2-1}/2} = \frac{x}{\sqrt{4x^2-1}}.)$$

Now we can substitute for y and calculate $\dfrac{d\mathcal{A}}{dx}$: $\mathcal{A} = \dfrac{1}{2}\dfrac{x^2}{\sqrt{4x^2-1}} \quad\Rightarrow$

$\dfrac{d\mathcal{A}}{dx} = \dfrac{1}{2}\left[\dfrac{\sqrt{4x^2-1}\,(2x) - x^2\left(\frac{1}{2}\right)(4x^2-1)^{-1/2}(8x)}{4x^2-1}\right]$. This is 0 when $2x\sqrt{4x^2-1} - 4x^3(4x^2-1)^{-1/2} = 0$

$\Leftrightarrow \quad 2x(4x^2-1)^{-1/2}\left[(4x^2-1) - 2x^2\right] = 0 \quad\Rightarrow\quad (4x^2-1) - 2x^2 = 0 \;\; (x > 0) \quad\Leftrightarrow\quad 2x^2 = 1 \quad\Rightarrow$

$x = \frac{1}{\sqrt{2}}$. So this is one possible value for an extremum. We must also test the endpoints of the interval over which x ranges. The largest value that x can attain is 1, and the smallest value of x occurs when $y = 1 \quad\Leftrightarrow$

$1 = x/\sqrt{4x^2-1} \quad\Leftrightarrow\quad x^2 = 4x^2 - 1 \quad\Leftrightarrow\quad 3x^2 = 1 \quad\Leftrightarrow\quad x = \frac{1}{\sqrt{3}}$. This will give the same value of $\mathcal{A}$ as will $x = 1$, since the geometric situation is the same (reflected through the line $y = x$). We calculate

$\mathcal{A}\left(\frac{1}{\sqrt{2}}\right) = \dfrac{1}{2}\dfrac{(1/\sqrt{2})^2}{\sqrt{4(1/\sqrt{2})^2 - 1}} = \dfrac{1}{4} = 0.25$, and $\mathcal{A}(1) = \dfrac{1}{2}\dfrac{1^2}{\sqrt{4(1)^2-1}} = \dfrac{1}{2\sqrt{3}} \approx 0.29$. So the maximum area is $\mathcal{A}(1) = \mathcal{A}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2\sqrt{3}}$ and the minimum area is $\mathcal{A}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}$.

Another method: Use the angle θ (see diagram above) as a variable:

$\mathcal{A} = \frac{1}{2}xy = \frac{1}{2}\left(\frac{1}{2}\sec\theta\right)\left(\frac{1}{2}\csc\theta\right) = \dfrac{1}{8\sin\theta\,\cos\theta} = \dfrac{1}{4\sin 2\theta}$. $\mathcal{A}$ is minimized when $\sin 2\theta$ is maximal, that is, when $\sin 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$. Also note that $A'E = x = \frac{1}{2}\sec\theta \le 1 \Rightarrow \sec\theta \le 2 \Rightarrow$ $\cos\theta \ge \frac{1}{2} \Rightarrow \theta \le \frac{\pi}{3}$, and similarly, $A'F = y = \frac{1}{2}\csc\theta \le 1 \Rightarrow \csc\theta \le 2 \Rightarrow \sin\theta \le \frac{1}{2} \Rightarrow \theta \ge \frac{\pi}{6}$.

As above, we find that $\mathcal{A}$ is maximized at these endpoints: $\mathcal{A}\left(\frac{\pi}{6}\right) = \dfrac{1}{4\sin\frac{\pi}{3}} = \dfrac{1}{2\sqrt{3}} = \dfrac{1}{4\sin\frac{2\pi}{3}} = \mathcal{A}\left(\frac{\pi}{3}\right)$; and minimized at $\theta = \frac{\pi}{4}$: $\mathcal{A}\left(\frac{\pi}{4}\right) = \dfrac{1}{4\sin\frac{\pi}{2}} = \dfrac{1}{4}$.

15. Suppose that the curve $y = a^x$ intersects the line $y = x$. Then $a^{x_0} = x_0$ for some $x_0 > 0$, and hence $a = x_0^{1/x_0}$. We find the maximum value of $g(x) = x^{1/x}$, > 0, because if a is larger than the maximum value of this function, then the curve $y = a^x$ does not intersect the line $y = x$.

$g'(x) = e^{(1/x)\ln x}\left(-\dfrac{1}{x^2}\ln x + \dfrac{1}{x}\cdot\dfrac{1}{x}\right) = x^{1/x}\left(\dfrac{1}{x^2}\right)(1 - \ln x)$. This is 0 only where $x = e$, and for $0 < x < e$, $f'(x) > 0$, while for $x > e$, $f'(x) < 0$, so g has an absolute maximum of $g(e) = e^{1/e}$. So if $y = a^x$ intersects $y = x$, we must have $0 < a \le e^{1/e}$. Conversely, suppose that $0 < a \le e^{1/e}$. Then $a^e \le e$, so the graph of $y = a^x$ lies below or touches the graph of $y = x$ at $x = e$. Also $a^0 = 1 > 0$, so the graph of $y = a^x$ lies above that of $y = x$ at $x = 0$. Therefore, by the Intermediate Value Theorem, the graphs of $y = a^x$ and $y = x$ must intersect somewhere between $x = 0$ and $x = e$.

16. If $L = \lim\limits_{x\to\infty}\left(\dfrac{x+a}{x-a}\right)^x$, then L has the indeterminate form 1^∞, so

$$\begin{aligned}
\ln L &= \lim_{x\to\infty}\ln\left(\frac{x+a}{x-a}\right)^x = \lim_{x\to\infty} x\ln\left(\frac{x+a}{x-a}\right) = \lim_{x\to\infty}\frac{\ln(x+a)-\ln(x-a)}{1/x} \\
&\overset{\text{H}}{=} \lim_{x\to\infty}\frac{\dfrac{1}{x+a}-\dfrac{1}{x-a}}{-1/x^2} = \lim_{x\to\infty}\left[\frac{(x-a)-(x+a)}{(x+a)(x-a)}\cdot\frac{-x^2}{1}\right] \\
&= \lim_{x\to\infty}\frac{2ax^2}{x^2-a^2} = \lim_{x\to\infty}\frac{2a}{1-a^2/x^2} = 2a.
\end{aligned}$$

Hence, $\ln L = 2a$, so $L = e^{2a}$. From the original equation, we want $L = e^1 \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$.

17. Note that $f(0) = 0$, so for $x \ne 0$, $\left|\dfrac{f(x)-f(0)}{x-0}\right| = \left|\dfrac{f(x)}{x}\right| = \dfrac{|f(x)|}{|x|} \le \dfrac{|\sin x|}{|x|} = \dfrac{\sin x}{x}$.

Therefore, $|f'(0)| = \left|\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}\right| = \lim\limits_{x\to 0}\left|\dfrac{f(x)-f(0)}{x-0}\right| \le \lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$. But $f'(x) = a_1\cos x + 2a_2\cos 2x + \cdots + na_n\cos nx$, so $|f'(0)| = |a_1 + 2a_2 + \cdots + na_n| \le 1$.

Another solution: We are given that $\left|\sum_{k=1}^n a_k\sin kx\right| \le |\sin x|$. So for x close to 0, and $x \ne 0$, we have $\left|\sum\limits_{k=1}^n a_k\dfrac{\sin kx}{\sin x}\right| \le 1 \Rightarrow \lim\limits_{x\to 0}\left|\sum\limits_{k=1}^n a_k\dfrac{\sin kx}{\sin x}\right| \le 1 \Rightarrow \left|\sum\limits_{k=1}^n a_k\lim\limits_{x\to 0}\dfrac{\sin kx}{\sin x}\right| \le 1$. But by l'Hospital's Rule, $\lim\limits_{x\to 0}\dfrac{\sin kx}{\sin x} = \lim\limits_{x\to 0}\dfrac{k\cos kx}{\cos x} = k$, so $\left|\sum\limits_{k=1}^n ka_k\right| \le 1$.

18. Let the circle have radius r, so $|OP| = |OQ| = r$, where O is the center of the circle. Now $\angle POR$ has measure $\frac{1}{2}\theta$, and $\angle OPR$ is a right angle, so $\tan\frac{1}{2}\theta = \dfrac{|PR|}{r}$ and the area of $\triangle OPR$ is $\frac{1}{2}|OP|\,|PR| = \frac{1}{2}r^2\tan\frac{1}{2}\theta$. The area of the sector cut by OP and OR is $\frac{1}{2}r^2\left(\frac{1}{2}\theta\right) = \frac{1}{4}r^2\theta$. Let S be the intersection of PQ and OR. Then $\sin\frac{1}{2}\theta = \dfrac{|PS|}{r}$ and $\cos\frac{1}{2}\theta = \dfrac{|OS|}{r}$, and the area of $\triangle OSP$ is

$\frac{1}{2}|OS|\,|PS| = \frac{1}{2}\left(r\cos\frac{1}{2}\theta\right)\left(r\sin\frac{1}{2}\theta\right) = \frac{1}{2}r^2\sin\frac{1}{2}\theta\,\cos\frac{1}{2}\theta = \frac{1}{4}r^2\sin\theta$.

So $B(\theta) = 2\left(\frac{1}{2}r^2\tan\frac{1}{2}\theta - \frac{1}{4}r^2\theta\right) = r^2\left(\tan\frac{1}{2}\theta - \frac{1}{2}\theta\right)$ and $A(\theta) = 2\left(\frac{1}{4}r^2\theta - \frac{1}{4}r^2\sin\theta\right) = \frac{1}{2}r^2(\theta - \sin\theta)$.

$$\text{Thus, }\lim_{\theta\to0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to0^+}\frac{\frac{1}{2}r^2(\theta-\sin\theta)}{r^2\left(\tan\frac{1}{2}\theta-\frac{1}{2}\theta\right)} = \lim_{\theta\to0^+}\frac{\theta-\sin\theta}{2\left(\tan\frac{1}{2}\theta-\frac{1}{2}\theta\right)} \overset{\text{H}}{=} \lim_{\theta\to0^+}\frac{1-\cos\theta}{2\left(\frac{1}{2}\sec^2\frac{1}{2}\theta-\frac{1}{2}\right)}$$

$$= \lim_{\theta\to0^+}\frac{1-\cos\theta}{\sec^2\frac{1}{2}\theta-1} = \lim_{\theta\to0^+}\frac{1-\cos\theta}{\tan^2\frac{1}{2}\theta} \overset{\text{H}}{=} \lim_{\theta\to0^+}\frac{\sin\theta}{2\left(\tan\frac{1}{2}\theta\right)\left(\sec^2\frac{1}{2}\theta\right)\frac{1}{2}}$$

$$= \lim_{\theta\to0^+}\frac{\sin\theta\cos^3\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = \lim_{\theta\to0^+}\frac{\left(2\sin\frac{1}{2}\theta\,\cos\frac{1}{2}\theta\right)\cos^3\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = 2\lim_{\theta\to0^+}\cos^4\left(\tfrac{1}{2}\theta\right) = 2(1)^4 = 2$$

19. (a) Distance $=$ rate $\times$ time, so time $=$ distance/rate. $T_1 = \dfrac{D}{c_1}$,

$$T_2 = \frac{2\,|PR|}{c_1} + \frac{|RS|}{c_2} = \frac{2h\sec\theta}{c_1} + \frac{D-2h\tan\theta}{c_2},\quad T_3 = \frac{2\sqrt{h^2+D^2/4}}{c_1} = \frac{\sqrt{4h^2+D^2}}{c_1}.$$

(b) $\dfrac{dT_2}{d\theta} = \dfrac{2h}{c_1}\cdot\sec\theta\,\tan\theta - \dfrac{2h}{c_2}\sec^2\theta = 0$ when $2h\sec\theta\left(\dfrac{1}{c_1}\tan\theta - \dfrac{1}{c_2}\sec\theta\right) = 0 \quad\Rightarrow$

$\dfrac{1}{c_1}\dfrac{\sin\theta}{\cos\theta} - \dfrac{1}{c_2}\dfrac{1}{\cos\theta} = 0 \quad\Rightarrow\quad \dfrac{\sin\theta}{c_1\cos\theta} = \dfrac{1}{c_2\cos\theta} \quad\Rightarrow\quad \sin\theta = \dfrac{c_1}{c_2}$. The First Derivative Test shows that this gives a minimum.

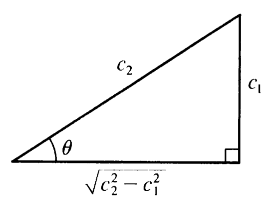
(c) Using part (a) with $D = 1$ and $T_1 = 0.26$, we have $T_1 = \dfrac{D}{c_1} \quad\Rightarrow$

$c_1 = \frac{1}{0.26} \approx 3.85$ km/s. $T_3 = \dfrac{\sqrt{4h^2+D^2}}{c_1} \quad\Rightarrow\quad 4h^2 + D^2 = T_3^2c_1^2 \quad\Rightarrow$

$h = \frac{1}{2}\sqrt{T_3^2c_1^2 - D^2} = \frac{1}{2}\sqrt{(0.34)^2(1/0.26)^2 - 1^2} \approx 0.42$ km. To find c_2, we use $\sin\theta = \dfrac{c_1}{c_2}$ from part (b) and $T_2 = \dfrac{2h\sec\theta}{c_1} + \dfrac{D-2h\tan\theta}{c_2}$ from part (a). From the figure,

$\sin\theta = \dfrac{c_1}{c_2} \quad\Rightarrow\quad \sec\theta = \dfrac{c_2}{\sqrt{c_2^2-c_1^2}}$ and $\tan\theta = \dfrac{c_1}{\sqrt{c_2^2-c_1^2}}$, so

$$T_2 = \frac{2hc_2}{c_1\sqrt{c_2^2-c_1^2}} + \frac{D\sqrt{c_2^2-c_1^2}-2hc_1}{c_2\sqrt{c_2^2-c_1^2}}.$$



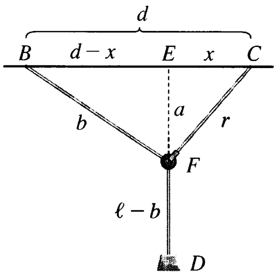
Using the values for T_2 [given as 0.32], h, c_1, and D, we can graph

$\mathrm{Y}_1 = T_2$ and $\mathrm{Y}_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2-c_1^2}} + \dfrac{D\sqrt{c_2^2-c_1^2}-2hc_1}{c_2\sqrt{c_2^2-c_1^2}}$ and find their intersection points. Doing so gives us $c_2 \approx 4.10$ and 7.66, but if $c_2 = 4.10$, then $\theta = \arcsin(c_1/c_2) \approx 69.6^\circ$, which implies that point S is to the left of point R in the diagram. So $c_2 = 7.66$ km/s.

20. A straight line intersects the curve $y = f(x) = x^4 + cx^3 + 12x^2 - 5x + 2$ in four distinct points if and only if the graph of f has two inflection points. $f'(x) = 4x^3 + 3cx^2 + 24x - 5$ and $f''(x) = 12x^2 + 6cx + 24$.

$f''(x) = 0 \quad \Leftrightarrow \quad x = \dfrac{-6c \pm \sqrt{(6c)^2 - 4(12)(24)}}{2(12)}$. There are two distinct roots for $f''(x) = 0$ (and hence two inflection points) if and only if the discriminant is positive; that is, $36c^2 - 1152 > 0 \quad \Leftrightarrow \quad c^2 > 32 \quad \Leftrightarrow$ $|c| > \sqrt{32}$. Thus, the desired values of c are $c < -4\sqrt{2}$ or $c > 4\sqrt{2}$.

21.



Let $a = |EF|$ and $b = |BF|$ as shown in the figure. Since $\ell = |BF| + |FD|$, $|FD| = \ell - b$. Now

$$\begin{aligned}|ED| &= |EF| + |FD| = a + \ell - b \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + a^2} \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + \left(\sqrt{r^2 - x^2}\right)^2} \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 - 2dx + x^2 + r^2 - x^2}\end{aligned}$$

Let $f(x) = \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 + r^2 - 2dx}$.

$f'(x) = \frac{1}{2}\left(r^2 - x^2\right)^{-1/2}(-2x) - \frac{1}{2}\left(d^2 + r^2 - 2dx\right)^{-1/2}(-2d) = \dfrac{-x}{\sqrt{r^2 - x^2}} + \dfrac{d}{\sqrt{d^2 + r^2 - 2dx}}$.

$f'(x) = 0 \quad \Rightarrow \quad \dfrac{x}{\sqrt{r^2 - x^2}} = \dfrac{d}{\sqrt{d^2 + r^2 - 2dx}} \quad \Rightarrow \quad \dfrac{x^2}{r^2 - x^2} = \dfrac{d^2}{d^2 + r^2 - 2dx} \quad \Rightarrow$

$d^2x^2 + r^2x^2 - 2dx^3 = d^2r^2 - d^2x^2 \quad \Rightarrow \quad 0 = 2dx^3 - 2d^2x^2 - r^2x^2 + d^2r^2 \quad \Rightarrow$

$0 = 2dx^2(x - d) - r^2\left(x^2 - d^2\right) \quad \Rightarrow \quad 0 = 2dx^2(x - d) - r^2(x + d)(x - d) \quad \Rightarrow$

$0 = (x - d)\left[2dx^2 - r^2(x + d)\right]$

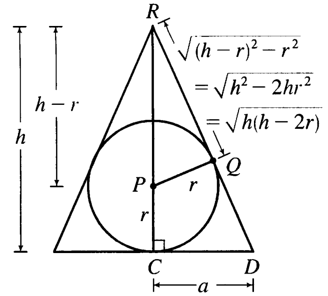
But $d > r > x$, so $x \neq d$. Thus, we solve $2dx^2 - r^2x - dr^2 = 0$ for x:

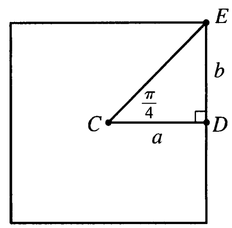
$x = \dfrac{-\left(-r^2\right) \pm \sqrt{\left(-r^2\right)^2 - 4(2d)\left(-dr^2\right)}}{2(2d)} = \dfrac{r^2 \pm \sqrt{r^4 + 8d^2r^2}}{4d}$. Because $\sqrt{r^4 + 8d^2r^2} > r^2$, the "negative" can be discarded. Thus,

$$\begin{aligned}x &= \frac{r^2 + \sqrt{r^2}\sqrt{r^2 + 8d^2}}{4d} = \frac{r^2 + r\sqrt{r^2 + 8d^2}}{4d} \qquad (r > 0) \\ &= \frac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)\end{aligned}$$

The maximum value of $|ED|$ occurs at this value of x.

22.





Let $a = \overline{CD}$ denote the distance from the center C of the base to the midpoint D of a side of the base.

Since ΔPQR is similar to ΔDCR, $\dfrac{a}{h} = \dfrac{r}{\sqrt{h(h-2r)}} \quad \Rightarrow \quad a = \dfrac{rh}{\sqrt{h(h-2r)}} = r\dfrac{\sqrt{h}}{\sqrt{h-2r}}$.

Let b denote one-half the length of a side of the base. The area A of the base is

$A = 8(\text{area of } \Delta CDE) = 8\left(\frac{1}{2}ab\right) = 4a\left(a\tan\frac{\pi}{4}\right) = 4a^2$.

The volume of the pyramid is $V = \frac{1}{3}Ah = \frac{1}{3}\left(4a^2\right)h = \dfrac{4}{3}\left(r\dfrac{\sqrt{h}}{\sqrt{h-2r}}\right)^2 h = \frac{4}{3}r^2\dfrac{h^2}{h-2r}$, with domain $h > 2r$.

Now $\dfrac{dV}{dh} = \dfrac{4}{3}r^2 \cdot \dfrac{(h-2r)(2h)-h^2(1)}{(h-2r)^2} = \dfrac{4}{3}r^2\dfrac{h^2-4hr}{(h-2r)^2} = \dfrac{4}{3}r^2\dfrac{h(h-4r)}{(h-2r)^2}$

and

$$\begin{aligned}\frac{d^2V}{dh^2} &= \frac{4}{3}r^2 \cdot \frac{(h-2r)^2(2h-4r)-\left(h^2-4hr\right)(2)(h-2r)(1)}{[(h-2r)^2]^2} \\ &= \frac{4}{3}r^2 \cdot \frac{2(h-2r)\left[\left(h^2-4hr+4r^2\right)-\left(h^2-4hr\right)\right]}{(h-2r)^2} \\ &= \frac{8}{3}r^2 \cdot \frac{4r^2}{(h-2r)^3} = \frac{32}{3}r^4 \cdot \frac{1}{(h-2r)^3}.\end{aligned}$$

The first derivative is equal to zero for $h = 4r$ and the second derivative is positive for $h > 2r$, so the volume of the pyramid is minimized when $h = 4r$.

To extend our solution to a regular n-gon, we make the following changes:

(1) the number of sides of the base is n

(2) the number of triangles in the base is $2n$

(3) $\angle DCE = \dfrac{\pi}{n}$

(4) $b = a\tan\dfrac{\pi}{n}$

We then obtain the following results:

$A = na^2\tan\dfrac{\pi}{n}$, $V = \dfrac{nr^2}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{h^2}{h-2r}$, $\dfrac{dV}{dh} = \dfrac{nr^2}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{h(h-4r)}{(h-2r)^2}$, and

$\dfrac{d^2V}{dh^2} = \dfrac{8nr^4}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{1}{(h-2r)^3}$. Notice that the answer, $h = 4r$, is independent of the number of sides of the base of the polygon!

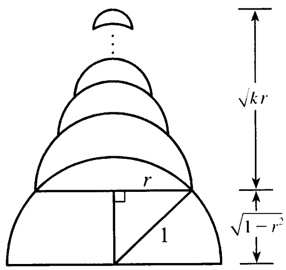
23. $V = \frac{4}{3}\pi r^3 \quad \Leftrightarrow \quad \dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$. But $\dfrac{dV}{dt}$ is proportional to the surface area, so $\dfrac{dV}{dt} = k \cdot 4\pi r^2$ for some constant k. Therefore, $4\pi r^2 \dfrac{dr}{dt} = k \cdot 4\pi r^2 \quad \Leftrightarrow \quad \dfrac{dr}{dt} = k =$ constant. An antiderivative of k with respect to t is kt, so $r = kt + C$. When $t = 0$, the radius r must equal the original radius r_0, so $C = r_0$, and $r = kt + r_0$. To find k we use the fact that when $t = 3$, $r = 3k + r_0$ and $V = \frac{1}{2}V_0 \quad \Rightarrow \quad \frac{4}{3}\pi(3k + r_0)^3 = \frac{1}{2} \cdot \frac{4}{3}\pi r_0^3 \quad \Rightarrow$

$(3k + r_0)^3 = \frac{1}{2}r_0^3 \quad \Rightarrow \quad 3k + r_0 = \frac{1}{\sqrt[3]{2}}r_0 \quad \Rightarrow \quad k = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)$. Since $r = kt + r_0$,

$r = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0$. When the snowball has melted completely we have $r = 0 \quad \Rightarrow$

$\frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0 = 0$ which gives $t = \dfrac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1}$. Hence, it takes $\dfrac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1} - 3 = \dfrac{3}{\sqrt[3]{2} - 1} \approx 11$ h 33 min longer.

24. By ignoring the bottom hemisphere of the initial spherical bubble, we can rephrase the problem as follows: Prove that the maximum height of a stack of n hemispherical bubbles is $\sqrt{n}$ if the radius of the bottom hemisphere is 1. We proceed by induction. The case $n = 1$ is obvious since $\sqrt{1}$ is the height of the first hemisphere. Suppose the assertion is true for $n = k$ and let's suppose we have $k + 1$ hemispherical bubbles forming a stack of maximum height. Suppose the second hemisphere (counting from the bottom) has radius r. Then by our induction hypothesis (scaled to the setting of a bottom hemisphere of radius r), the height of the stack formed by the top k bubbles is $\sqrt{k}\,r$. (If it were shorter, then the total stack of $k + 1$ bubbles wouldn't have maximum height.)



The height of the whole stack is $H(r) = \sqrt{k}\,r + \sqrt{1 - r^2}$. (See the figure.) We want to choose r so as to maximize $H(r)$. Note that $0 < r < 1$. We calculate

$H'(r) = \sqrt{k} - \dfrac{r}{\sqrt{1 - r^2}}$ and $H''(r) = \dfrac{-1}{(1 - r^2)^{3/2}}$. $H'(r) = 0 \quad \Leftrightarrow$

$r^2 = k(1 - r^2) \quad \Leftrightarrow \quad (k + 1)r^2 = k \quad \Leftrightarrow \quad r = \sqrt{\dfrac{k}{k + 1}}$. This is the only critical number in $(0, 1)$ and it represents a local maximum (hence an absolute maximum) since $H''(r) < 0$ on $(0, 1)$. When $r = \sqrt{\dfrac{k}{k + 1}}$,

$H(r) = \sqrt{k}\dfrac{\sqrt{k}}{\sqrt{k + 1}} + \sqrt{1 - \dfrac{k}{k + 1}} = \dfrac{k}{\sqrt{k + 1}} + \dfrac{1}{\sqrt{k + 1}} = \sqrt{k + 1}$. Thus, the assertion is true for $n = k + 1$ when it is true for $n = k$. By induction, it is true for all positive integers n.

Note: In general, a maximally tall stack of n hemispherical bubbles consists of bubbles with radii $1, \sqrt{\dfrac{n - 1}{n}}, \sqrt{\dfrac{n - 2}{n}}, \ldots, \sqrt{\dfrac{2}{n}}, \sqrt{\dfrac{1}{n}}$.